Partial Differential Equations (TATA27)
Spring Semester 2017
Homework 8

8.1 Derive the mean value property for harmonic functions of three variables (Theorem 5.6 with $n = 3$) by viewing harmonic functions u as solutions to the wave equation (6.10) which are independent of t. [Hint: In the notation of Section 6.5.1, the mean value of u around $\mathbf{x}$ is $\overline{u}_{\mathbf{x}}(r,t)$, but we may as well write this as $\overline{u}_{\mathbf{x}}(r)$ if it is independent of t. The mean value property is then the equality $\lim_{s\to 0} \overline{u}_{\mathbf{x}}(s) = \overline{u}_{\mathbf{x}}(r)$.]

8.2 Show that radial solutions u to the wave equation (6.10) take the form

$$u(x,y,z,t) = \frac{f(r+t) + g(r-t)}{r},$$

where $r = \sqrt{x^2+y^2+z^2}$ and f and g are arbitrary twice differentiable functions. [Hint: If u is radial then $u(x,y,z,t) = \overline{u}_{\mathbf{0}}(r,t)$.]

8.3 Use (9.3) to prove that

$$\int_{-\infty}^{\infty} H_k(x) H_\ell(x) e^{-x^2}\, dx = 0$$

for $k \neq \ell$.

8.4 (a) Show that if w solves (9.4) with $\lambda = 2n+1$, then $x \mapsto 2xw(x) - w'(x)$ solves (9.4) with $\lambda = 2n+3$.

(b) Show that the alternative expression for the Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n(e^{-x^2})}{dx^n}$$

indeed solves (9.4) with $\lambda = 2n+1$ for $n = 0, 1, 2, \ldots$. [Hint: You could, for example, use 8.4(a) and induction on n.]