Partial Differential Equations (TATA27)
Spring Semester 2017
Homework 5

5.1 Solve (6.1) with:

(a) $g(x) = e^x$ and $h(x) = \sin x$;

(b) $g(x) = \log(1 + x^2)$ and $h(x) = 4 + x$.

5.2 Suppose both g and h are odd functions and u is the solution of (6.1). Show that $u(\cdot, t)$ is also odd for each $t > 0$.

5.3 By factorising the operator as we did in Section 6.1, solve the following initial value problems.

(a)

$$\begin{cases} \partial_{tt}u(x,t) - 3\partial_{xt}u(x,t) - 4\partial_{xx}u(x,t) = 0 & \text{for } x \in \mathbf{R} \text{ and } t > 0, \\ u(x,0) = x^2 \quad \text{and} \quad \partial_t u(x,0) = e^x & \text{for } x \in \mathbf{R}. \end{cases}$$

(b)

$$\begin{cases} \partial_{tt}u(x,t) + \partial_{xt}u(x,t) - 20\partial_{xx}u(x,t) = 0 & \text{for } x \in \mathbf{R} \text{ and } t > 0, \\ u(x,0) = x^2 \quad \text{and} \quad \partial_t u(x,0) = e^x & \text{for } x \in \mathbf{R}. \end{cases}$$

5.4 For a solution u of the wave equation $\partial_{tt}u(x,t) - \partial_{xx}u(x,t) = 0$ (with $\rho = T = c = 1$, $x \in \mathbf{R}$ and $t > 0$), the energy density is defined to be

$$e(x,t) = \frac{1}{2}((\partial_t u(x,t))^2 + (\partial_x u(x,t))^2)$$

and the momentum density

$$p(x,t) = \partial_t u(x,t)\partial_x u(x,t).$$

(a) Show that $\partial e/\partial t = \partial p/\partial x$ and $\partial p/\partial t = \partial e/\partial x$.

(b) Show that e and p also satsify the wave equation.

5.5 Suppose that u is a solution of the wave equation $\partial_{tt}u(x,t) - c^2\partial_{xx}u(x,t) = 0$ ($x \in \mathbf{R}$, $t > 0$).

(a) Show that for a fixed $y \in \mathbf{R}$, v defined by $v(x,t) = u(x - y, t)$ is also a solution of the wave equation.

(b) Show that for a fixed $a \in \mathbf{R}$, w defined by $w(x,t) = u(ax, at)$ is also a solution of the wave equation.

5.6 Consider a solution u to the *damped string equation*

$$\partial_{tt}u(x,t) - c^2\partial_{xx}u(x,t) + r\partial_t u(x,t) = 0 \quad (x \in \mathbf{R},\ t > 0)$$

for $c^2 = T/\rho$ and $T, \rho, r > 0$. Define the energy by the same formula we used in class:

$$E[u](t) = \frac{1}{2}\int_{-\infty}^{\infty} \rho(\partial_t u(x,t))^2 + T(\partial_x u(x,t))^2 dx.$$

Assuming u and its derivatives are sufficiently smooth and decay as $x \to \pm\infty$, show that the energy $E[u]$ is a non-increasing function.