Partial Differential Equations (TATA27)
Spring Semester 2017
Homework 4

4.1 Let Ω be an open set with C^1 boundary. For $\lambda \geq 0$, define the energy of each continuously differentiable $v\colon \overline{\Omega} \to \mathbf{R}$ to be

$$E_\lambda[v] = \frac{1}{2}\int_\Omega (|\nabla v(\mathbf{x})|^2 + \lambda |v(\mathbf{x})|^2) d\mathbf{x}.$$

Show that a function $u \in C^2(\overline{\Omega})$ which satisfies $\Delta u - \lambda u = 0$ in Ω is such that

$$E_\lambda[u] \leq E_\lambda[v]$$

for all $v \in C^1(\overline{\Omega})$ such that $v(\mathbf{x}) = u(\mathbf{x})$ for all $\mathbf{x} \in \partial\Omega$.

Observe that the energy $E_\lambda[v]$ makes sense for functions in $C^1(\overline{\Omega})$, but (assuming a solution to the corresponding boundary value problem exists) a minimiser can sometimes be found in a better class. For example, if $\lambda = 0$, Lemma 5.5 tells us any solution u is smooth in Ω.

4.2 Let Ω be an open set with C^1 boundary and $h\colon \partial\Omega \to \mathbf{R}$ a C^1 function. Define the energy of each continuously differentiable $v\colon \Omega \to \mathbf{R}$ to be

$$E_h[v] = \frac{1}{2}\int_\Omega |\nabla v(\mathbf{x})|^2 d\mathbf{x} - \int_{\partial\Omega} h(\mathbf{x})v(\mathbf{x})d\sigma(\mathbf{x}).$$

Show that a function $u \in C^2(\overline{\Omega})$ which satisfies the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \text{ and} \\ \frac{\partial u}{\partial \mathbf{n}} := \mathbf{n}\cdot\nabla u = h & \text{on } \partial\Omega \end{cases}$$

is such that

$$E_h[u] \leq E_h[v]$$

for all $v \in C^1(\overline{\Omega})$. Here $\mathbf{n}$ is the outward unit normal to $\partial\Omega$.

Here, in contrast to question 4.1, the boundary condition $\partial u/\partial\mathbf{n} = h$ is incorporated into the energy and we see that a solution u is a minimum of E_h over all $v \in C^1(\overline{\Omega})$ regardless of how v behaves at the boundary.

4.3 The aim of this question is to prove Theorem 5.12. Let Ω be an open bounded set with C^2 boundary.

(a) In this part of the question we will prove that the Green's function for the Laplacian in Ω is unique. Suppose we have two Green's functions G_1 and G_2 for the Laplacian in Ω.

i. For each fixed $\mathbf{x} \in \Omega$, prove that $\mathbf{y} \mapsto G_1(\mathbf{x},\mathbf{y}) - G_2(\mathbf{x},\mathbf{y})$ has a continuous extension which belongs to $C^2(\overline{\Omega})$ and is harmonic in Ω.

ii. By considering a boundary value problem that $\mathbf{y} \mapsto G_1(\mathbf{x},\mathbf{y}) - G_2(\mathbf{x},\mathbf{y})$ solves, prove that $G_1 = G_2$.

(b) We now wish to prove the Green's function is symmetric, that is $G(\mathbf{x},\mathbf{y}) = G(\mathbf{y},\mathbf{x})$ for all $\mathbf{x},\mathbf{y} \in \Omega$.

i. Fix $\mathbf{x},\mathbf{y} \in \Omega$ with $\mathbf{x} \neq \mathbf{y}$ and consider the functions $\mathbf{z} \mapsto u(\mathbf{z}) := G(\mathbf{x},\mathbf{z})$ and $\mathbf{z} \mapsto v(\mathbf{z}) := G(\mathbf{y},\mathbf{z})$. Apply Green's second identity (5.10) to u and v in the domain $\Omega_r := \Omega \setminus \overline{(B_r(\mathbf{x}) \cup B_r(\mathbf{y}))}$ for $r > 0$ so small that $\overline{(B_r(\mathbf{x}) \cup B_r(\mathbf{y}))} \subset \Omega$ and $\overline{B_r(\mathbf{x})} \cap \overline{B_r(\mathbf{y})} = \emptyset$ to obtain that

$$\begin{aligned} 0 = &\int_{\partial B_r(\mathbf{x})} G(\mathbf{x},\mathbf{z})\frac{\partial G(\mathbf{y},\cdot)}{\partial\mathbf{n}}(\mathbf{z}) - G(\mathbf{y},\mathbf{z})\frac{\partial G(\mathbf{x},\cdot)}{\partial\mathbf{n}}(\mathbf{z})d\sigma(\mathbf{z}) \\ &+ \int_{\partial B_r(\mathbf{y})} G(\mathbf{x},\mathbf{z})\frac{\partial G(\mathbf{y},\cdot)}{\partial\mathbf{n}}(\mathbf{z}) - G(\mathbf{y},\mathbf{z})\frac{\partial G(\mathbf{x},\cdot)}{\partial\mathbf{n}}(\mathbf{z})d\sigma(\mathbf{z}). \end{aligned} \tag{$\dagger$}$$

ii. Using the definition of the Green's function, prove that

$$\int_{\partial B_r(\mathbf{x})} (G(\mathbf{x},\mathbf{z}) - \Phi(\mathbf{z}-\mathbf{x})) \frac{\partial G(\mathbf{y},\cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) \to 0$$

and

$$\int_{\partial B_r(\mathbf{x})} G(\mathbf{y},\mathbf{z}) \left(\frac{\partial G(\mathbf{x},\cdot)}{\partial \mathbf{n}}(\mathbf{z}) - \frac{\partial \Phi}{\partial \mathbf{n}}(\mathbf{z}-\mathbf{x}) \right) d\sigma(\mathbf{z}) \to 0$$

as $r \to 0$.

iii. Using the same ideas as in the proof of Lemma 5.9 prove that

$$\int_{\partial B_r(\mathbf{x})} \Phi(\mathbf{z}-\mathbf{x}) \frac{\partial G(\mathbf{y},\cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) = 0$$

and

$$\int_{\partial B_r(\mathbf{x})} G(\mathbf{y},\mathbf{z}) \frac{\partial \Phi}{\partial \mathbf{n}}(\mathbf{z}-\mathbf{x}) d\sigma(\mathbf{z}) = -G(\mathbf{y},\mathbf{x}).$$

iv. Combine the results above to show that

$$\int_{\partial B_r(\mathbf{x})} G(\mathbf{x},\mathbf{z}) \frac{\partial G(\mathbf{y},\cdot)}{\partial \mathbf{n}}(\mathbf{z}) - G(\mathbf{y},\mathbf{z}) \frac{\partial G(\mathbf{x},\cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) \to G(\mathbf{y},\mathbf{x}). \tag{‡}$$

as $r \to 0$. (Observe the left-hand side of (‡) is the first term on the right-hand side of (†).)

v. Swap the roles of $\mathbf{x}$ and $\mathbf{y}$ in (‡) to conclude a similar statement for the second term on the right-hand side of (†). Combine your answer with (†) and (‡) to prove $G(\mathbf{x},\mathbf{y}) = G(\mathbf{y},\mathbf{x})$.

4.4 Prove the following lemma, which is a generalisation of Lemma 5.9 that does not assume that u is harmonic.

Lemma. *Let Ω be an open bounded set with C^1 boundary and suppose that $u \in C^2(\overline{\Omega})$ is such that $\Delta u = f$ for some $f \in C(\overline{\Omega})$. Then*

$$u(\mathbf{x}) = \int_{\partial\Omega} \left\{ \Phi(\mathbf{y}-\mathbf{x}) \left(\frac{\partial u}{\partial \mathbf{n}} \right)(\mathbf{y}) - \left(\frac{\partial \Phi}{\partial \mathbf{n}} \right)(\mathbf{y}-\mathbf{x}) u(\mathbf{y}) \right\} d\sigma(\mathbf{y}) - \int_\Omega f(\mathbf{y}) \Phi(\mathbf{y}-\mathbf{x}) d\mathbf{y}.$$

for each $\mathbf{x} \in \Omega$.

[Hint: Follow the proof of Lemma 5.9.]

4.5 Use the lemma from question 4.4 to prove the following generalisation of Theorem 5.11.

Theorem. *Let $\Omega \subset \mathbf{R}^n$ be an open bounded set with C^2 boundary, and suppose $h \in C^2(\partial\Omega)$ and $f \in C(\overline{\Omega})$. If G is a Green's function for the Laplacian in Ω then the solution of the boundary value problem*

$$\begin{cases} \Delta u = f & \text{in } \Omega, \text{ and} \\ u = h & \text{on } \partial\Omega, \end{cases} \tag{$*$}$$

is given by

$$u(\mathbf{x}) = -\int_{\partial\Omega} \left(\frac{\partial G(\mathbf{x},\cdot)}{\partial \mathbf{n}}(\mathbf{y}) \right) h(\mathbf{y}) d\sigma(\mathbf{y}) + \int_\Omega f(\mathbf{x}) G(\mathbf{x},\mathbf{y}) d\mathbf{y}.$$

where $(\partial G(\mathbf{x},\cdot)/\partial \mathbf{n})(\mathbf{y}) := \mathbf{n}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} G(\mathbf{x},\mathbf{y})$ is the normal derivative of $\mathbf{y} \mapsto G(\mathbf{x},\mathbf{y})$.

We proved the uniqueness of solutions to $(*)$ in Section 5.2 of our notes, so when we can find a Green's function we have both the existence and uniqueness of solutions to $(*)$.