Partial Differential Equations (TATA27)
Spring Semester 2017
Homework 2

2.1 Let $\mathbf{R}^2_+ = \mathbf{R} \times (0, \infty)$, $C(\overline{\mathbf{R}^2_+})$ denote the set of continuous real-valued functions on $\overline{\mathbf{R}^2_+}$ and $C^1(\mathbf{R}^2_+)$ denote the set of continuously differentiable real-valued functions on $\mathbf{R}^2_+$. Consider the boundary-value problem

$$\begin{cases} u_x(x,y) + yu_y(x,y) = 0 & \text{for all } (x,y) \in \mathbf{R}^2_+, \text{ and} \\ u(x,0) = \phi(x) & \text{for all } x \in \mathbf{R}. \end{cases}$$

(a) Show that if $\phi(x) = x$ for all $x \in \mathbf{R}$, then no solution exists in $C(\overline{\mathbf{R}^2_+}) \cap C^1(\mathbf{R}^2_+)$.

(b) Show that if $\phi(x) = 1$ for all $x \in \mathbf{R}$, then there are many solutions in $C(\overline{\mathbf{R}^2_+}) \cap C^1(\mathbf{R}^2_+)$.

2.2 Fix $\ell > 0$ and consider the following boundary-value problem. Given a function $f\colon (0,\ell) \to \mathbf{R}$ we wish to find $u\colon [0,\ell] \to \mathbf{R}$ which is twice continuously differentiable such that

$$\begin{cases} u''(x) + u'(x) = f(x) & \text{for all } x \in (0,\ell), \text{ and} \\ u'(0) = u(0) = \frac{1}{2}(u'(\ell) + u(\ell)). \end{cases}$$

(a) Prove that if a solution u exists, it is not unique.

(b) Find two conditions we must place on f for a solution to exist.

2.3 Suppose $u\colon \mathbf{R}^2 \to \mathbf{R}$ is a harmonic function.

(a) For constants $a, b \in \mathbf{R}$ show that $v\colon \mathbf{R}^2 \to \mathbf{R}$ defined by

$$v(x,y) = u(x+a, y+b) \quad \text{for all } x, y \in \mathbf{R}$$

is harmonic.

(b) For a constant $\alpha \in \mathbf{R}$ show that $w\colon \mathbf{R}^2 \to \mathbf{R}$ defined by

$$w(x,y) = u(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha) \quad \text{for all } x, y \in \mathbf{R}$$

is harmonic.

This exercise shows that Laplace's equation in the plane is invariant under rigid motions (translations and rotations).

2.4 Let Ω be a bounded open set. Prove that functions $u\colon \overline{\Omega} \to \mathbf{R}$ which satisfy

$$\Delta u(\mathbf{x}) + \mathbf{x} \cdot \nabla u(\mathbf{x}) \geq 0$$

for $\mathbf{x} \in \Omega$ also satisfy the weak maximum principle:

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

2.5 The Schrödinger equation is a good model for the behaviour of particles at the atomic and subatomic level. Solutions $u\colon \mathbf{R}^3 \times \mathbf{R} \to \mathbf{C}$ are complex-valued and are related to the probability that a particle can be found in a specific region. The equation which models the motion of an electron around a hydrogen nucleus has the form

$$-i\hbar\frac{\partial u}{\partial t}(\mathbf{x},t) = \frac{\hbar^2}{2m}\Delta u(\mathbf{x},t) + \frac{e^2}{|\mathbf{x}|}u(\mathbf{x},t)$$

for real constants $\hbar$, m and e and all $\mathbf{x} \in \mathbf{R}^3$ and $t \in \mathbf{R}$. Assume that u and $\partial_t u$ are continuous functions, and u, $\partial_t u$ and ∇u satisfy the estimate $|u(\mathbf{x},t)|^2 + |\partial_t u(\mathbf{x},t)|^2 + |\nabla u(\mathbf{x},t)|^2 \leq C(1+|\mathbf{x}|)^{-3-\varepsilon}$ for some $C, \varepsilon > 0$, so we can interchange integration and differentiation according to the formula

$$\frac{d}{dt}\int_{\mathbf{R}^3} u(\mathbf{x},t)d\mathbf{x} = \int_{\mathbf{R}^3} \partial_t u(\mathbf{x},t)d\mathbf{x}$$

(see *Strauss*, p. 420, for the result for one spatial variable, but the same rule applies in $\mathbf{R}^3$). Show that if

$$\int_{\mathbf{R}^3} |u(\mathbf{x},t_0)|^2 d\mathbf{x} = 1$$

for some $t_0 \in \mathbf{R}$, then

$$\int_{\mathbf{R}^3} |u(\mathbf{x},t)|^2 d\mathbf{x} = 1$$

for all $t \in \mathbf{R}$.