Partial Differential Equations (TATA27)
Spring Semester 2017
Homework 1

1.1 Prove that the following operators are linear operators.

(a) $\nabla = (\partial_1, \partial_2, \ldots, \partial_n)$ acting on functions $u\colon \mathbf{R}^n \to \mathbf{R}$.

(b) The divergence operator div which acts via the formula $\operatorname{div}(u) = \sum_{j=1}^n \partial_j u^j$ on functions $u = (u^1, u^2, \ldots, u^n)\colon \mathbf{R}^n \to \mathbf{R}^n$.

(c) curl acting on functions $u = (u_1, u_2, u_3)\colon \mathbf{R}^3 \to \mathbf{R}^3$ by the formula

$$\operatorname{curl}(u) = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1).$$

(d) $\Delta := \nabla \cdot \nabla = \sum_{j=1}^n \partial_j^2$ acting on functions $u\colon \mathbf{R}^n \to \mathbf{R}$.

1.2 Classify the following equations in u as linear or non-linear (non-linear means not linear) and give the order of the equation.

(a) $u_{tt}(x,t) - u_{xx}(x,t) + xu(x,t) = 0$

(b) $u_{tt}(x,t) - u_{xx}(x,t) + x^2 = 0$

(c) $u_t(x,t) + u_{xxxx}(x,t) + \sqrt{1 + u(x,t)} = 0$

(d) $u_x(x,y) + e^y u_y(x,y) = 0$

1.3 Use the method of characteristics to find an explicit formula for a smooth function $u\colon \mathbf{R}^2 \to \mathbf{R}$ which solves the equation

$$u_x(x,y) + yu_y(x,y) = 0 \quad \text{for all } x, y \in \mathbf{R}$$

and satisfies the condition $u(0,y) = g(y)$ for all $y \in \mathbf{R}$ where g is a given smooth function.

1.4 Use the method of characteristics to find an explicit formula for a smooth function $u\colon \mathbf{R}^2 \to \mathbf{R}$ which solves the equation

$$(1 + x^2)u_x(x,y) + u_y(x,y) = 0 \quad \text{for all } x, y \in \mathbf{R}$$

and satisfies the condition $u(0,y) = g(y)$ for all $y \in \mathbf{R}$ where g is a given smooth function.

1.5 Let $f\colon \mathbf{R}^n \times (0,\infty) \to \mathbf{R}$ and $g\colon \mathbf{R}^n \to \mathbf{R}$ be two smooth functions and $b \in \mathbf{R}^n$. Consider the equations

$$\begin{aligned} u_t(x,t) + b \cdot \nabla u(x,t) &= f(x,t) \quad \text{for } x \in \mathbf{R}^n \text{ and } t > 0, \text{ and} \\ u(x,0) &= g(x) \quad \text{for } x \in \mathbf{R}^n. \end{aligned} \tag{$\dagger$}$$

Here ∇ denotes the gradient vector in the x-variables. Set $z(s) = u(x + bs, t + s)$ for fixed $x \in \mathbf{R}^n$ and $t > 0$ and derive an ODE which z satisfies. Use this ODE to find a formula for a solution u to ($\dagger$). (This method simply takes the characteristic curves (X,T) to be $X(s) = x + bs$ and $T(s) = t + s$.)

1.6 Let a, b and c be real numbers and suppose that $b \neq 0$. Use the method of characteristics to find an explicit formula for a smooth function $u\colon \mathbf{R}^2 \to \mathbf{R}$ which solves the equation

$$au_x(x,t) + bu_t(x,t) + cu(x,t) = 0 \quad \text{for all } x \in \mathbf{R} \text{ and } t > 0$$

and satisfies the "initial condition" $u(x,0) = g(x)$ for all $x \in \mathbf{R}$ where g is a given smooth function.