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Svar till Tentamen TATA 27 Partial Differential Equations
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1.
$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^1 \Big(e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt} \Big) y^2 dy,$$
$$u_x(0,t) = \frac{1}{\sqrt{4\pi kt}} \frac{1}{kt} \int_0^{1/4kt} z e^{-z} dz = \frac{4h^{-1/2}}{\pi} \Big(1 - (1+h)e^{-h} \Big)$$
where $h = 1/4kt$. By differentiating one can check that the right hand side is a positive and increasing function of h.

2.
$$u(x,y) = \sum_{k=0}^{\infty} (a_k \sin(k+1/2)t + b_k \cos(k+1/2)t) \sin(k+1/2)x.$$
using the initial conditions, we obtain
$$u(x,y) = 2\sin(t/2)\sin(x/2) + \sum_{k=0}^{\infty} b_k \cos(k+1/2)t \sin(k+1/2)x,$$
where
$$b_k = \frac{4}{\pi(2k+1)}.$$

3. Consider the function $v(x,y) = u(x,y) - (x^2+y^2+1)/4$. This function is harmonic and $v(x,y) = \cos^2(x+y) - 1/2$ for $x^2+y^2 = 1$. Observing that $cos^2(x+y) - 1/2 = \cos(2(x+y))/2$, we conclude that $|v| \le 1/2$ for $x^2+y^2 = 1$. Application of the maximum principle leads to the required inequality.

4. The Euler equation is
$$2u + u' = \frac{d}{dx}(u + 4u') \quad \text{or} \quad u'' = \frac{1}{2}u.$$
Its general solution is
$$u = ae^{-x/\sqrt{2}} + be^{x/\sqrt{2}}.$$
To find a and b we use the boundary conditions:
$$u(2) = ae^{-\sqrt{2}} + be^{\sqrt{2}} = 2 \text{ and } u(4) = ae^{-2\sqrt{2}} + be^{2\sqrt{2}} = 3.$$

Solving this system, we find

$$a = \frac{3 - 2e^{\sqrt{2}}}{1 - e^{-2\sqrt{2}}} \text{ and } b = \frac{3 - 2e^{-\sqrt{2}}}{e^{2\sqrt{2}} - 1}$$

Finally,

$$\min = b^2\Big(2+\frac{1}{\sqrt{2}}\Big)\frac{1}{\sqrt{2}}\Big(e^{4\sqrt{2}}-e^{2\sqrt{2}}\Big)+a^2\Big(2-\frac{1}{\sqrt{2}}\Big)\frac{1}{\sqrt{2}}\Big(e^{-2\sqrt{2}}-e^{-4\sqrt{2}}\Big)$$

5.

$$u(x) = \frac{1}{4}\Big(e^{-2|x-3|} - e^{-2(x+3)}\Big).$$

6. Let u_1 and u_2 be two solutions. Then the function $u = u_1 - u_2$ satisfies

$$u_t - \Delta u = 0 \quad \text{in } D \text{ for } t > 0 \tag{1}$$

and

$$u = 0 \text{ for } x^2 + y^2 + z^2 = 1, \quad \frac{\partial u}{\partial \hat{n}} = 0 \text{ for } x^2 + y^2 + z^2 = 5, \text{ for } t > 0.$$

Moreover $u = 0$ for $t = 0$. Multiplying (1) by u and integrating over D, we obtain

$$\int_D (u_t u - \Delta u\, u) dxdydz = 0.$$

Using Green's first identity for the second term in the left-hand side and taking into account the homogeneous boundary conditions, we arrive at

$$\int_D (u_t u + \nabla u \cdot \nabla u) dxdydz = 0.$$

Since $2u_t u = d(u^2)/dt$, we can rewrite it as

$$\int_D \Big(\frac{1}{2}\frac{d}{dt}u^2 + |\nabla u|^2\Big) dxdydz = 0.$$

Integrating this from 0 to T we obtain

$$\int_D (u^2(x, y, z, T) - u^2(x, y, z, 0)) dxdydz + \int_0^T \int_D |\nabla u|^2 dxdydz = 0.$$

Since $u = 0$ for $t = 0$, we have $\nabla u = 0$ and $u = 0$ for $t = T$. Since T is arbitrary, this implies $u = 0$ and hence $u_1 = u_2$.