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**Tentamen TATA 30 Partial Differential Equations and Finite Elements 20 October 2006, 08-13**

You can use on this examination Formelsamlingen i partiella diff.ekv. för M3 av L-E Andersson. No calculators.

1. Consider the following Neumann problem for the wave equation on the half-line:

$$u_{tt} - 4u_{xx} = 0 \quad \text{for } t > 0 \text{ and } 0 < x < \infty$$

and

$$u_x(0,t) = 0 \quad \text{for } t > 0,$$

supplied with the initial conditions

$$u(x,0) = 0 \quad \text{and} \quad u_t(x,0) = \begin{cases} x-1 & \text{for } 1 \le x \le 2 \\ 0 & \text{for } 0 \le x < 1 \text{ and for } x > 2. \end{cases}$$

Find $u(1, 5/4)$.

2. Prove the maximum principle for the equation

$$u_{xx} + u_{yy} + 2u_x = 0 \quad \text{on } D,$$

where $D$ is the rectangle $(0,1) \times (0,2)$. Namely, prove that

$$\max_{\overline{D}} u(x,y) \le \max_{bdyD} u(x,y).$$

Hint: examine the function $u(x,y) + \varepsilon(y-c)^2$ with an appropriate choice of $\varepsilon$ and $c$.

3. Solve the following initial boundary value problem for the heat equation:

$$u_t - 5u_{xx} = 0 \quad \text{for } t > 0 \text{ and } 1 < x < 2,$$

$$u(1,t) = u_x(2,t) = 0 \quad \text{for } t > 0$$

and

$$u(x,0) = x - 1 \quad \text{for } 1 < x < 2.$$

4. Solve the following problem:

$$u_{tt} = u_{xx} + 2u_x + u \quad \text{for } t > 0 \text{ and } -\infty < x < \infty$$

and

$$u(x,0) = \delta(x), \quad u_t(x,0) = 0 \quad \text{for } -\infty < x < \infty.$$

5. Let $g$ be a distribution on $\mathbb{R}$. Consider the following equation

$$f' = g, \tag{1}$$

where $f$ is unknown distribution.

(i) Let $\psi \in \mathcal{D}(\mathbb{R})$ satisfy

$$\int_{-\infty}^{\infty} \psi(x)dx = 1. \tag{2}$$

Show that the functional

$$(f, \phi) = -(g, \int_{-\infty}^{x} \phi(y)dy - \int_{-\infty}^{\infty} \phi(y)dy \int_{-\infty}^{x} \psi(y)dy) \tag{3}$$

is a distribution and solves equation (1).

(ii) Let $\psi_1$ and $\psi_2$ be two test functions satisfying (2) and let $f_1$ and $f_2$ be the distributions given by (3). Show that $f_1 - f_2 = \text{const}$.

6. Let $D$ be a bounded domain in $\mathbb{R}^3$ with boundary $bdyD$. Prove that

$$\min \int\int\int_D (|\nabla u|^2 - 2fu)dxdydz,$$

where min is taken over all $u$ which is zero on $bdyD$, is attained on the solution of

$$-\Delta u = f \quad \text{in } D$$

and

$$u = 0 \quad \text{on } bdyD.$$