LINKÖPINGS TEKNISKA HÖGSKOLA
Matematiska institutionen
Vladimir Kozlov

Svar till Tentamen TATA 30 Partial Differential Equations and Finite Elements 20 October 2006, 08-13

1. $\frac{5}{32}$

2. Suppose that u attains its max at an interior point $(x_0, y_0) \in D$ and

$$M = u(x_0, y_0) > m = \max_{bdyD} u(x, y).$$

Let $v(x, y) = u(x, y) + \varepsilon(y - y_0)^2$. Then $v(x_0, y_0) = M$ and

$$v(x, y) \leq u(x, y) + \varepsilon \max_{bdyD}(y - y_0)^2 \leq m + 4\varepsilon < M \quad \text{on } bdyD,$$

provided $0 < \varepsilon < (M - m)/4$. Therefore, v attains its max at an interior point $(x_1, y_1) \in D$. We have

$$v_{xx}(x_1, y_1) + v_{yy}(x_1, y_1) + 2v_x(x_1, y_1) = 2\varepsilon. \tag{1}$$

But since (x_1, y_1) is a max-point for v, we have $v_x(x_1, y_1) = 0$ and $v_{xx}(x_1, y_1) \leq 0$, $v_{yy}(x_1, y_1) \leq 0$. These contradict to (1).

3.
$$u(x, t) = \sum_{k=0}^{\infty} \frac{2 \sin \beta_k}{\beta_k^2} e^{-5\beta_k^2 t} \sin \beta_k (x - 1),$$

where $\beta_k = \frac{\pi}{2} + k\pi$.

4.
$$u(x, t) = \frac{1}{2}\Big(e^t \delta(x + t) + e^{-t} \delta(x - t)\Big).$$

5. (i) The function

$$\int_{-\infty}^{x} \phi(y)dy - \int_{-\infty}^{\infty} \phi(y)dy \int_{-\infty}^{x} \psi(y)dy$$

belongs to $\mathcal{D}$ since it is zero for large x. Therefore formula (3) defines a distribution.

(ii) We have

$$(f', \phi) = -(f, \phi') = (g, \int_{-\infty}^{x} \phi'(y)dy - \int_{-\infty}^{\infty} \phi'(y)dy \int_{-\infty}^{x} \psi(y)dy) = (g, \phi),$$

which proves that $f' = g$ for such choice of f.

(ii) Let ψ_1 and ψ_2 be two test functions satisfying (2) and let f_1 and f_2 be the distributions given by (3). Then

$$(f_1 - f_2, \phi) = (g, \int_{-\infty}^{\infty} \phi(y)dy \int_{-\infty}^{x} (\psi_1 - \psi_2)(y)dy).$$

Therefore

$$(f_1 - f_2, \phi) = c \int_{-\infty}^{\infty} \phi(y)dy$$

with

$$c = (g, \int_{-\infty}^{x} (\psi_1 - \psi_2)(y)dy).$$

This proves that $f_1 - f_2 = c$.

6. Let

$$I(v) = \int\int\int_D (|\nabla v|^2 - 2fv)dxdydz$$

and let u satisfy: $-\Delta u = f$ in D and $u = 0$ on $bdyD$. Let us show that $I(v) > I(u)$ for $v \neq u$ and $v = 0$ on $bdyD$. We represent v as $v = u + w$. Then

$$I(v) = I(u + w) = I(u) + I(w) + 2 \int\int\int_D (\nabla u \cdot \nabla w)dxdydz.$$

Using the first Green formula and that $w = 0$ on $bdyD$ we obtain

$$\int\int\int_D (\nabla u \cdot \nabla w)dxdydz = -\int\int\int_D (\Delta u\, wdxdydz = \int\int\int_D fwdxdydz.$$

Therefore

$$I(v) = I(u+w) = I(u)+I(w)+2\int\int\int_D fwdxdydz = I(u)+\int\int\int_D |\nabla w|^2 dxdydz$$

This proves that $I(v) > I(u)$ for $v \neq u$.