Lösningsskisser till tentamen i TATA27 Partiella differentialekvationer 2013-05-27

1. Using the formula sheet, the correct ansatz with Neumann boundary conditions is $u(x,t) = a_0(t)/2 + \sum_{k=1}^{\infty} a_k(t)\cos(\frac{\pi}{2}kx)$. The PDE gives the ODEs $a_0'(t) = 0$, $a_k'(t) = -3(\pi k/2)^2 a_k(t)$, $k \geq 1$, for the coefficients. Thus $u(x,t) = a_0/2 + \sum_{k=1}^{\infty} a_k \exp(-3(\pi k/2)^2 t)\cos(\frac{\pi}{2}kx)$. The inital data give

$$a_k = \int_0^2 x^2 \cos(\pi kx/2)dx = (2/(k\pi))^3 \int_0^{\pi k} t^2 \cos t dt = \ldots = \frac{16(-1)^k}{\pi^2 k^2}, \qquad k \geq 1,$$

by making the change of variables $t = k\pi x/2$ when $k \geq 1$. We obtain the solution

$$u(x,t) = \frac{4}{3} + \frac{16}{\pi^2}\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} e^{-3(\frac{\pi}{2}k)^2 t}\cos(\tfrac{\pi}{2}kx),$$

where we see that $\lim_{t\to\infty} u = 4/3$.

2. (a) Using the change of variables $\xi = x - ct, \eta = x + ct$, the chain rule shows that the wave equation is equivalent to

$$c^2 u_{\xi\xi} + c^2 u_{\eta\eta} - 2c^2 u_{\xi\eta} = c^2(u_{\xi\xi} + u_{\eta\eta} + 2u_{\xi\eta}),$$

that is $u_{\xi\eta} = 0$. Integration gives the general solution $u(\xi,\eta) = f(\xi) + g(\eta)$ as claimed.

(b) The boundary and initial conditions read

$$\begin{cases} f(-ct) + g(ct) = b(t), & t > 0, \\ f(x) + g(x) = a(x), & x > 0, \\ -cf'(x) + cg'(x) = 0, & x > 0. \end{cases}$$

The last equation shows that $f(s) = g(s) + d$ for all $s > 0$, where d is some constant. Thus the second gives $g(s) = (a(s) - d)/2$, $f(s) = (a(s) + d)/2$ for $s > 0$, and the first gives $f(-s) = b(s/c) - g(s)$ for $s > 0$. Thus gives the solution

$$u(x,t) = \begin{cases} b(t - x/c) + \frac{1}{2}(a(x + ct) - a(ct - x)), & 0 < x < ct, \\ \frac{1}{2}(a(x + ct) + a(x - ct)), & 0 < ct < x. \end{cases}$$

Note that we do not need to know $g(s)$ for $s < 0$, or the value of the constant d.

3. (a) From the formula sheet, the fundamental solution for the plane is $\frac{1}{4\pi}\ln(x^2 + y^2)$. Applying the method of reflection, we see that

$$\begin{aligned} G_D = \tfrac{1}{4\pi}\ln((x-a)^2 + (y-b)^2) - \tfrac{1}{4\pi}\ln((x+a)^2 + (y-b)^2) \\ + \tfrac{1}{4\pi}\ln((x+a)^2 + (y+b)^2) - \tfrac{1}{4\pi}\ln((x-a)^2 + (y+b)^2) \end{aligned}$$

is zero on ∂D and differs from the fundamental solution, centered at (a,b), by terms which are harmonic in D. Thus this is indeed the Green's function.

(b) By Green's formulas $u(a,b) = \int_{\partial D} \frac{\partial G_D}{\partial n} u ds$. On the positive x-axis, we have

$$\frac{\partial G_D}{\partial n} = -\partial_y G_D((x,y),(a,b))|_{y=0} = -\frac{1}{4\pi}\Big(\frac{-4b}{(x-a)^2+b^2} + \frac{4b}{(x+a)^2+b^2}\Big).$$

This gives the solution

$$u(a,b) = \frac{b}{\pi}\int_0^\infty \left(\frac{1}{(x-a)^2+b^2} - \frac{1}{(x+a)^2+b^2}\right) h(x)dx.$$

4. In polar coordinates, we have $f_1 = 1-r$, $f_2 = (r-r^2)\cos\varphi$, $\nabla f_1 = -\hat{r}$ and $\nabla f_2 = (1-2r)\cos\varphi\hat{r} - (1-r)\sin\varphi\hat{\varphi}$, and get

$$\int_D f_1^2 = 2\pi\int_0^1 (r-2r^2+r^3)dr = \pi/6, \qquad \int_D f_2^2 = \pi\int_0^1 (r^3-2r^4+r^5)dr = \pi/60,$$
$$\int_D |\nabla f_1|^2 = 2\pi\int_0^1 rdr = \pi, \qquad \int_D |\nabla f_2|^2 = \pi\int_0^1 (2r-6r^2+5r^3)dr = \pi/4.$$

On the other hand, it is clear that $\int_D f_1 f_2 dxdy = \int_D \nabla f_1 \cdot \nabla f_2 dxdy = 0$ since $\int_0^{2\pi} \cos\varphi d\varphi = 0$. Therefore the matrices in the Rayleigh–Ritz method are diagonal and we get the Rayleigh quotients $\pi/(\pi/6) = 6$ and $(\pi/4)/(\pi/60) = 15$ as approximations to the first two eigenvalues. (More exact $\lambda_1 \approx 5.78$ and $\lambda_2 \approx 14.68$.)

5. (a) This is the Laplace equation, and by the maximum principle for this equation, $u(x_0,y_0) \geq 0$ at some boundary point. $E = \partial D$ is the smallest possible such set.

   (b) This is the wave equation. Such a point need not exist, which the example $u(x,y) = 2\sin(x)\sin(y) - 1$ shows. (A full proof!)

   (c) With $t = x$, this is the backward heat equation. The maximum principle for the heat equation, with time reversed, tells that if $u \leq 1/2$ on $E := \partial D \cap \{(x,y)\ ;\ x \geq 1/2\}$, then $u(1/2,1/2) \leq 1/2$. Therefore, if now $u(1/2,1/2) = 1$, then $\max_E u \geq 0$.

6. Differentiate twice:

$$f'(r) = \frac{1}{2\pi}\int_0^{2\pi} \big(\cos\varphi u'_x(x+r\cos\varphi, y+r\sin\varphi) + \sin\varphi u'_y(x+r\cos\varphi, y+r\sin\varphi)\big)\, d\varphi,$$
$$f''(r) = \frac{1}{2\pi}\int_0^{2\pi} (\cos^2\varphi u'_{xx}(x+r\cos\varphi, y+r\sin\varphi)$$
$$+ \sin(2\varphi)u'_{xy}(x+r\cos\varphi, y+r\sin\varphi) + \sin^2\varphi u'_{yy}(x+r\cos\varphi, y+r\sin\varphi))d\varphi.$$

Using the continuity of the second derivatives, we get

$$0 = \lim_{r\to 0} f''(r) = \frac{1}{2\pi}\int_0^{2\pi} (\cos^2\varphi u'_{xx}(x,y) + \sin(2\varphi)u'_{xy}(x,y) + \sin^2\varphi u'_{yy}(x,y))d\varphi$$
$$= \tfrac{1}{2}u_{xx}(x,y) + \tfrac{1}{2}u_{yy}(x,y),$$

which proves that u is a harmonic function.