LINKÖPINGS TEKNISKA HÖGSKOLA
Matematiska institutionen
Vladimir Kozlov

Svar till Tentamen TATA 27 Partial Differential Equations
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1.
$$u(x,t) = 1 + \frac{1}{\sqrt{8\pi t}} \int_0^\infty \left(e^{-(x-y)^2/8t} - e^{-(x+y)^2/8t}\right)(y^2-1)dy.$$

Therefore
$$u_x(0,t) = \frac{1}{2\sqrt{8\pi t}} \int_0^\infty e^{-y^2/8t} y(y^2-1)dy$$

After the change of variable $y^2 = 8tz$ we obtain
$$u_x(0,t) = \sqrt{\frac{2t}{\pi}} \int_0^\infty e^{-z}(8tz-1)dz = \sqrt{\frac{2t}{\pi}}(8t-1)$$

This implies that $u_x(0,1) > 0$.

2.
$$u(x,t) = 1 - \frac{x}{3} + v(x,t),$$

where v satisfies the problem
$$v_{tt} - 3v_{xx} = 0 \quad \text{for } 0 < t \text{ and } 0 < x < 3,$$
$$v(0,t) = 0 \text{ and } v(3,t) = 0 \text{ for } 0 < t$$

and
$$v(x,0) = \frac{x}{3} - 1, \quad v_t(x,0) = 0 \quad \text{for } 0 < x < 3.$$

Now one can use the method of separation of variables. The answer is
$$u(x,t) = 1 - \frac{x}{3} + \sum_{n=1}^\infty A_n \cos(\sqrt{3}\beta_n t)\,\sin(\beta_n x),$$

where
$$A_n = -\frac{2}{3\beta_n}, \quad \beta_n = \frac{n\pi}{3}.$$

3. Let $u(x,y) = x^2/2 - 2x + v(x,y)$. Then
$$v_{xx} + v_{yy} = 0 \text{ for } 0 < x < 4 \text{ and } 0 < y < 2$$

and satisfies the Dirichlet boundary conditions
$$v(0,y) = v(4,y) = 0 \quad \text{for } 0 < y < 2$$

together with

$$v(x,0) = 1 + 2x - x^2/2 \quad \text{and} \quad v(x,2) = 2 + 2x - x^2/2.$$

Now applying maximum principle to v and observing that $0 \le v \le 4$ on the boundary we arrive at the required inequality.

4. Equation for u is $u'' - u - x = 0$. It's general solution is

$$u(x) = -x + Ae^x + Be^{-x}.$$

Using boundary conditions we obtain

$$A = 2(e^2 + e^{-2})^{-1}, \quad B = -A.$$

5. $f''(x) = 6\delta(x-1) + 6\delta(x+2) + h(x)$, where $h(x) = 2$ for $x < -2$ and for $x > 1$ and $h(x) = -2$ for $-2 \le x \le 1$.

6. The proof is essentially the same as for 4june 2009, problem 6.