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Tentamen TATA 27 Partial Differential Equations
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You can use on this examination Formelsamlingen i partiella diff.ekv. för M3 av L-E Andersson. No calculators.

1. Consider the following Dirichlet problem for the wave equation on the half-line:
$$u_{tt} - 3u_{xx} = 0 \quad \text{for } t > 0 \text{ and } 0 < x < \infty$$
and
$$u(0,t) = 0 \quad \text{for } t > 0,$$
satisfying the initial conditions $u(x,0) = 0$ for $x > 0$ and
$$u_t(x,0) = \begin{cases} 1 & \text{for } 1 \le x \le 2 \\ 0 & \text{for } x > 2 \text{ and for } 0 < x < 1. \end{cases}$$
Find the solution to this problem and calculate $u(1/2, 1/3)$.

2. Solve the following boundary value problem for the Laplace equation:
$$u_{xx} + u_{yy} = 0 \quad \text{for } 0 < x < 3 \text{ and } 0 < y < 2,$$
$$u_y(x,0) = u(x,2) = 0 \quad \text{for } 0 < x < 3$$
and
$$u_x(0,y) = y^2 - 4 \quad \text{and} \quad u(3,y) = 0 \quad \text{for } 0 < y < 2.$$

3. Consider the function $u = u(x,t)$ which satisfies the heat equation
$$u_t - u_{xx} = 1 \quad \text{for } 0 < x < 1 \text{ and } t > 0,$$
the Dirichlet conditions
$$u(0,t) = u(1,t) = 0) \quad \text{for } t > 0$$
and the initial condition $u(x,0) = 0$. Show that $x(1-x)/2 - 1/8 \le u(x,t) \le x(1-x)/2$ and then improve the lower estimate by showing that u is nonnegative.

4. Find
$$\min \int_0^1 (u^2 + 3u'^2 + xu')dx,$$
where the minimum is taken over all functions satisfying $u(0) = 1$ and $u(1) = 2$.

5. Let

$$f(x) = \begin{cases} x & \text{for } x < 0 \\ 1 & \text{for } 0 < x < 1 \\ x^3 & \text{for } x > 1 \end{cases}$$

Calculate the second derivative of f in distributional sense.

6. Show that the wave equation

$$u_{tt} - 4u_{xx} = \cos(e^t x) \quad \text{for } 0 < x < 1 \text{ and } t > 0$$

supplied with boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u_x(1,t) = 0 \quad \text{for } t > 0$$

and satisfying the initial conditions

$$u(x,0) = u_t(x,0) = 0 \quad \text{for } 0 < x < 1$$

has at most one solution. (Use the energy method for proving this uniqueness result).