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Tentamen TATA 27 Partial Differential Equations
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You can use on this examination Formelsamlingen i partiella diff.ekv. för M3 av L-E Andersson. No calculators.

1. Consider the following Dirichlet problem for the heat equation on the half-line:
$$u_t - ku_{xx} = 0 \quad \text{for } t > 0 \text{ and } 0 < x < \infty$$
and
$$u(0,t) = 0 \quad \text{for } t > 0,$$
supplied with the initial condition
$$u(x,0) = x^2 \quad \text{for } x > 0$$
Calculate explicitly the flux ku_x at $x = 0$ for all $t > 0$.

2. Solve the following boundary value problem for the wave equation:
$$u_{tt} = u_{xx} \quad \text{for } 0 < x < 1 \text{ and } 0 < t,$$
$$u(0,t) = u(1,t) = 0 \quad \text{for } 0 < x < 1$$
and
$$u(x,0) = \sin \pi x \quad \text{and} \quad u_t(x,0) = 1 \quad \text{for } 0 < x < 1.$$

3. Consider the function $u = u(x,t)$ which satisfies the Poisson equation
$$u_{xx} + u_{yy} = 1$$
on the unite square $0 < x < 1$ and $0 < y < 1$ and zero Dirichlet boundary conditions on the boundary of this unite square. Show by using the maximum principle that
$$u(1/2, 1/2) \leq -1/16$$
(Hint: use the maximum principle for the function $u - ((x-\alpha)^2 + (y-\alpha)^2)/4$ with a certain $\alpha \in [0,1]$).

4. Find
$$\min \int_0^\infty (u'^2 + xuu' + u^2)dx,$$
where the minimum is taken over all functions satisfying $u(0) = 1$ and $u(x) \to 0$ as $x \to \infty$.

5. Let

$$f(x) = \begin{cases} \sin x & \text{for } x < 0 \\ e^x - 1 & \text{for } 0 < x < 1 \\ x^3 & \text{for } x > 1 \end{cases}$$

Calculate the second derivative of f in distributional sense.

6. Let V be a bounded domain in $\mathbb{R}^3$ with boundary S. Consider the following boundary value problem

$$\begin{cases} \Delta u = f & \text{in } V \\ \frac{\partial u}{\partial \hat{n}} + \alpha u = 0 & \text{on } S, \end{cases}$$

where α is a positive constant, $\hat{n}$ is the outward unit normal to the boundary and f is a function on V. Prove that this problem has at most one solution.