Partial Differential Equations (TATA27)
Spring Semester 2015
Derivatives of Integrals

Theorem 1. *Suppose that* $f\colon [a,b]\times[c,d]\to\mathbf{R}$ *and* $\partial_t f$ *are continuous functions on* $[a,b]\times[c,d]$. *Then*

$$\frac{d}{dt}\left(\int_a^b f(x,t)dx\right)=\int_a^b \partial_t f(x,t)dx$$

for $t\in[c,d]$.

To prove this we first need to recall some things from Calculus. A function $f\colon D\to\mathbf{R}$ is said to be *continuous* if for each $x\in D$ and $\varepsilon>0$, there exists a $\delta=\delta(x,\varepsilon)>0$ such that

$$\text{if } y\in D \text{ is such that } |y-x|<\delta \text{ then } |f(y)-f(x)|<\varepsilon. \tag{1}$$

A function $f\colon D\to\mathbf{R}$ is said to be *uniformly continuous* if for each $\varepsilon>0$, there exists a $\delta=\delta(\varepsilon)>0$ such that

$$\text{if } x,y\in D \text{ are such that } |y-x|<\delta \text{ then } |f(y)-f(x)|<\varepsilon.$$

The difference between continuity and uniform continuity is that the δ in the definition of continuity may depend on both x and ε, but in uniform continuity it depends only on ε and not on x.

Lemma. *If* $f\colon C\to\mathbf{R}$ *is a continuous function on a compact set* C*, then* f *is uniformly continuous.*

Proof. Fix $\varepsilon>0$. We know that for each $x\in C$, there exists a $\delta=\delta(x,\varepsilon)>0$ such that (1) holds. Our task is for find a δ for which (1) holds for all $x\in C$.

Implication (1) can be rewritten as

$$f(B(x,\delta(x)))\subset B(f(x),\varepsilon). \tag{2}$$

The collection of balls $\{B(x,\delta(x))\}_{x\in C}$ is an open cover of C and equally so is $\{B(x,\delta(x)/2)\}_{x\in C}$. Thus, since C is compact, we can find a finite collection of $x_i\in D$, $i=1,2,\dots,N$ such that $\{B(x_i,\delta(x_i)/2)\}_{i=1}^N$ is also an open cover. Define $\delta=\min_{i=1,2,\dots,N}\delta(x_i)/2$. We will now show that this δ is the number we are seeking.

Then take arbitrary $x,y\in D$ which are such that $|x-y|<\delta$. Since $\{B(x_i,\delta(x_i)/2)\}_{i=1}^N$ is a cover of D, there must be at least one $k\in\{1,2,\dots,N\}$ such that $x\in B(x_k,\delta(x_k)/2)$. Consequently

$$|y-x_k|\le|y-x|+|x-x_k|<\delta+\delta(x_k)/2\le\delta(x_k)/2+\delta(x_k)/2=\delta(x_k),$$

so both x and y lie in $B(x_i,\delta(x_i))$. By (2), we then know that $|f(y)-f(x)|\le|f(y)-f(x_k)|+|f(x)-f(x_k)|<2\varepsilon$, as required. ☐

Proof of Theorem 1. By the linearity of the integral

$$\frac{1}{h}\left(\int_a^b f(x,t+h)dx-\int_a^b f(x,t)dx\right)=\int_a^b\frac{f(x,t+h)-f(x,t)}{h}dx.$$

The limit of the expression on the left as $h\to0$ is

$$\frac{d}{dt}\left(\int_a^b f(x,t)dx\right),$$

so we would like to show that the limit of the right is

$$\int_a^b \partial_2 f(x,t)dx.$$

By the mean value theorem, we know that for each x, t and h there exists a $\xi=\xi(x,t,h)$, which is between t and $t+h$, such that

$$\frac{f(x,t+h)-f(x,t)}{h}=\partial_2 f(x,\xi).$$

Since $\partial_2 f$ is continuous on the compact set $[a,b]\times[c,d]$ is it uniformly continuous. This means for a given $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon)>0$ such that (for any $s,t\in[c,d]$)

$$\text{if } s,t\in[c,d] \text{ are such that } |s-t|<\delta \text{ then } |\partial_2 f(x,s)-\partial_2 f(x,t)|<\varepsilon/(b-a).$$

Thus, for a given $\varepsilon>0$ we take $\delta>0$ as above and consider h such that $|h|<\delta$. Then we know that $|\xi(x,t,h)-t|\le|(t+h)-t|=|h|<\delta$ and

$$\begin{aligned}
&\left|\int_a^b \frac{f(x,t+h)-f(x,t)}{h}dx-\int_a^b \partial_2 f(x,t)dx\right|\\
&=\left|\int_a^b \partial_2 f(x,\xi(x,t,h))-\partial_2 f(x,t)dx\right|\\
&\le\int_a^b |\partial_2 f(x,\xi(x,t,h))-\partial_2 f(x,t)|dx\\
&\le\int_a^b \varepsilon/(b-a)dx=\varepsilon.
\end{aligned}$$

Since ε was arbitrary this proves that

$$\int_a^b \frac{f(x,t+h)-f(x,t)}{h}dx\to\int_a^b \partial_2 f(x,t)dx$$

as $h\to 0$. □

If f is defined on $[a,\infty)\times[c,d]$ we can consider the improper Riemann integrals

$$\int_a^\infty f(x,t)dx \tag{3}$$

for each $t\in[c,d]$. Recall that we say such an integral is *convergent* if the limit

$$\lim_{R\to\infty}\int_a^R f(x,t)dx.$$

exists and is finite. Suppose now the integral (3) is convergent for each t and let us denote the limit by $\ell(t)$. This means that for each $\varepsilon>0$ there exists a $N=N(t,\varepsilon)>0$ such that

$$\text{if } R>N \text{ then } \left|\textstyle\int_a^R f(x,t)dx-\ell(t)\right|<\varepsilon.$$

We say the integral *converges uniformly* in t if N depends only on ε and not on t (so we can write $N=N(\varepsilon)$).

Theorem 2. *Suppose that $f\colon[a,\infty)\times[c,d]\to\mathbf{R}$ and $\partial_t f$ are continuous functions on $[a,\infty)\times[c,d]$ and the integrals $\int_a^\infty|f(x,t)|dx$ and $\int_a^\infty|\partial_t f(x,t)|dx$ converge uniformly in $t\in[c,d]$. Then*

$$\frac{d}{dt}\left(\int_a^\infty f(x,t)dx\right)=\int_a^\infty \partial_t f(x,t)dx$$

for $t\in[c,d]$.

Proof. Define

$$I_n(t)=\int_a^n f(x,t)dx$$

for $n\in\mathbf{N}$. By Theorem 1 we know that each I_n is differentiable on $[c,d]$. Applying the Mean Value Theorem to I_n-I_m, we have

$$(I_n(t)-I_m(t))-(I_n(s)-I_m(s))=(t-s)(I_n'(\xi)-I_m'(\xi))$$

for some $\xi=\xi(s,t)$ between s and t. This implies

$$\frac{I_n(t)-I_n(s)}{t-s}-\frac{I_m(t)-I_m(s)}{t-s}=I_n'(\xi)-I_m'(\xi)=\int_m^n \partial_2 f(x,t)dx,$$

where in the last inequality we used Theorem 1. Since the integral $\int_a^\infty \partial_t f(x,t)dx$ converges uniformly in t, this means that for a given $\varepsilon > 0$ there exists an $N_1 = N_1(\varepsilon) > 0$ such that if $n, m \geq N_1$ then

$$\left|\frac{I_n(t) - I_n(s)}{t-s} - \frac{I_m(t) - I_m(s)}{t-s}\right| < \varepsilon.$$

Letting $m \to \infty$ we see that

$$\left|\frac{I_n(t) - I_n(s)}{t-s} - \frac{I(t) - I(s)}{t-s}\right| < \varepsilon. \tag{4}$$

for $n \geq N_1$, where

$$I(t) = \lim_{m\to\infty} I_m(t) = \int_a^\infty f(x,t)dx.$$

By the triangle inequality

$$\begin{aligned}
&\left|\frac{I(t) - I(s)}{t-s} - \int_a^\infty \partial_t f(x,t)dx\right| \\
&\leq \left|\frac{I(t) - I(s)}{t-s} - \frac{I_n(t) - I_n(s)}{t-s}\right| + \left|\frac{I_n(t) - I_n(s)}{t-s} - \int_a^n \partial_t f(x,t)dx\right| \\
&\quad + \left|\int_a^n \partial_t f(x,t)dx - \int_a^\infty \partial_t f(x,t)dx\right|
\end{aligned} \tag{5}$$

The first term on the right-hand side of (5) is less than ε if $n \geq N_1$ by (4). By Theorem 1 there exists a $\delta > 0$ such that the second term on the right-hand side of (5) will be less than ε if $|t-s| < \delta$. The third term on the right-hand side of (5) is equal to

$$\left|\int_n^\infty \partial_t f(x,t)dx\right| \leq \int_n^\infty |\partial_t f(x,t)|dx,$$

and since the integral of $\int_a^\infty |\partial_t f(x,t)|dx$ is convergent uniformly in t, there exists an $N_2 > 0$ such that $\int_n^\infty |\partial_t f(x,t)|dx$ is less than ε if $n \geq N_1$.

In conclusion, for a given $\varepsilon > 0$ we can take $\delta = \delta(\varepsilon) > 0$ as above and apply (5) with $n > \max\{N_1, N_2\}$, then

$$\text{if } |t-s| < \delta \text{ we can conclude that } \left|\tfrac{I(t)-I(s)}{t-s} - \int_a^\infty \partial_t f(x,t)dx\right| \leq 3\varepsilon,$$

which proves the theorem. □