Partial Differential Equations (TATA27)
Spring Semester 2019
Solutions to Homework 7

7.1 Recall that the solution to (6.1) is given by (6.3):

$$u(x,t) = \frac{1}{2}\left(g(x+ct)+g(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy.$$

(a) If we take $g(x) = e^x$ and $h(x) = \sin x$ then

$$u(x,t) = \frac{1}{2}\left(e^{x+ct}+e^{x-ct}\right) - \frac{1}{2c}\left(\cos(x+ct)-\cos(x-ct)\right).$$

(b) If we take $g(x) = \log(1+x^2)$ and $h(x) = 4+x$ then

$$u(x,t) = \frac{1}{2}\left(\log(x+ct)+\log(x-ct)\right) + \frac{4ct+xct}{c}.$$

7.2 Using (6.3), we have

$$u(-x,t) = \frac{1}{2}\left(g(-x+ct)+g(-x-ct)\right) + \frac{1}{2c}\int_{-x-ct}^{-x+ct} h(y)dy$$

But if g and h are odd functions then

$$\begin{aligned}
u(-x,t) &= -\frac{1}{2}\left(g(x-ct)+g(x+ct)\right) - \frac{1}{2c}\int_{-x-ct}^{-x+ct} h(-y)dy \\
&= -\frac{1}{2}\left(g(x+ct)+g(x-ct)\right) + \frac{1}{2c}\int_{x+ct}^{x-ct} h(z)dz \\
&= -\frac{1}{2}\left(g(x+ct)+g(x-ct)\right) - \frac{1}{2c}\int_{x-ct}^{x+ct} h(z)dz \\
&= -u(x,t),
\end{aligned}$$

so $u(\cdot,t)$ is an odd function for each t.

7.3 (a) We can factorise the differential equation as $(\partial_t+\partial_x)(\partial_t-4\partial_x)u(x,t) = 0$ and so can then view it as the system

$$\begin{cases} (\partial_t+\partial_x)v(x,t) = 0; \\ (\partial_t-4\partial_x)u(x,t) = v(x,t). \end{cases}$$

Using the method of characteristics, we can see that the general solution to the first equation is $v(x,t) = f(x-t)$ where f is any differentiable function.

To solve the second equation we first solve the homogeneous equation associated to it. Again, the method of characteristics shows the general solution to the homogeneous equation $(\partial_t-4\partial_x)u_h(x,t) = 0$ is $u_h(x,t) = g(x+4t)$ where g is any differentiable function. It is easy to find a particular solution to the second equation by starting with the anzats $u_p(x,t) = V(x-t)$. Substituting this into the equation gives

$$-V'(x-t) - 4V'(x-t) = f(x-t),$$

so a particular solution is $u_p(x,t) = V(x-t)$ where $V(y) = -(1/5)\int f(y)dy$. We add u_p and u_h together to obtain the general solution: $u(x,t) = V(x-t)+g(x+4t)$.

Now we can make use of the initial conditions to choose g and V. We see that we require

$$\begin{cases} u(x,0) = V(x)+g(x) = x^2; \\ \partial_t u(x,0) = -V'(x)+4g'(x) = e^x. \end{cases}$$

It is convenient to differentiate the first equation, so we obtain two conditions for V' and g':

$$\begin{cases} V'(x)+g'(x) = 2x; \\ -V'(x)+4g'(x) = e^x. \end{cases}$$

This is easy to solve and tells us that $V'(x) = (8x - e^x)/5$ and $g'(x) = (2x + e^x)/5$. This means that $V(x,t) = (4x^2 - e^x)/5 + C_1$ and $g(x) = (x^2 + e^x)/5 + C_1$ for some constants C_1 and C_2, but the intial condition $u(x,0) = x^2$ tells us that $C_1 = -C_2$, so

$$u(x,t) = \frac{4(x-t)^2 - e^{x-t} + (x+4t)^2 + e^{x+4t}}{5}$$

(b) We can factorise the differential equation as $(\partial_t - 4\partial_x)(\partial_t + 5\partial_x)u(x,t) = 0$ and so can then view it as the system

$$\begin{cases} (\partial_t - 4\partial_x)v(x,t) = 0; \\ (\partial_t + 5\partial_x)u(x,t) = v(x,t). \end{cases}$$

Using the method of characteristics just as before, we can see that the general solution to the first equation is $v(x,t) = f(x+4t)$ where f is any differentiable function.

To solve the second equation we first again solve the homogeneous equation associated to it. The method of characteristics shows the general solution to the homogeneous equation $(\partial_t + 5\partial_x)u_h(x,t) = 0$ is $u_h(x,t) = g(x-5t)$ where g is any differentiable function. A particular solution to the second equation is found by starting with the anzats $u_p(x,t) = V(x+4t)$. Substituting this into the equation gives

$$4V'(x+4t) + 5V'(x+4t) = f(x+4t),$$

so a particular solution is $u_p(x,t) = V(x+4t)$ where $V(y) = (1/9)\int f(y)dy$. We add u_p and u_h together to obtain the general solution: $u(x,t) = V(x+4t) + g(x-5t)$.

Now we can make use of the initial conditions to choose g and V. We see that we require

$$\begin{cases} u(x,0) = V(x) + g(x) = x^2; \\ \partial_t u(x,0) = 4V'(x) - 5g'(x) = e^x. \end{cases}$$

It is convenient to differentiate the first equation, so we obtain two conditions for V' and g':

$$\begin{cases} V'(x) + g'(x) = 2x; \\ 4V'(x) - 5g'(x) = e^x. \end{cases}$$

This tells us that $g'(x) = (8x - e^x)/9$ and $V'(x) = (10x + e^x)/9$. This means that $g(x,t) = (4x^2 - e^x)/9 + C_1$ and $V(x) = (5x^2 + e^x)/9 + C_2$ for some constants C_1 and C_2, but the intial condition $u(x,0) = x^2$ tells us that $C_1 = -C_2$, so

$$u(x,t) = \frac{4(x-5t)^2 - e^{x-5t} + 5(x+4t)^2 + e^{x+4t}}{9}$$

7.4 For a solution u of the wave equation $\partial_{tt}u - \partial_{xx}u = 0$ (with $\rho = T = c = 1$), the energy density is defined to be

$$e(x,t) = \frac{1}{2}((\partial_t u(x,t))^2 + (\partial_x u(x,t))^2)$$

and the momentum density

$$p(x,t) = \partial_t u(x,t)\partial_x u(x,t).$$

(a) We compute

$$\begin{aligned} \frac{\partial e}{\partial t}(x,t) &= \partial_t u(x,t)\partial_{tt}u(x,t) + \partial_x u(x,t)\partial_{xt}u(x,t) \\ \frac{\partial e}{\partial x}(x,t) &= \partial_t u(x,t)\partial_{tx}u(x,t) + \partial_x u(x,t)\partial_{xx}u(x,t) \\ \frac{\partial p}{\partial t}(x,t) &= \partial_{tt}u(x,t)\partial_x u(x,t) + \partial_t u(x,t)\partial_{xt}u(x,t) \\ \frac{\partial p}{\partial x}(x,t) &= \partial_{tx}u(x,t)\partial_x u(x,t) + \partial_t u(x,t)\partial_{xx}u(x,t) \end{aligned}$$

So clearly if $\partial_{tt}u - \partial_{xx}u = 0$ then $\partial e/\partial t = \partial p/\partial x$ and $\partial p/\partial t = \partial e/\partial x$.

(b) Furthermore, using the above results, if $\partial_{tt}u - \partial_{xx}u = 0$ we have

$$\frac{\partial^2 e}{\partial t^2}(x,t) = \frac{\partial^2 p}{\partial t \partial x}(x,t) = \frac{\partial^2 p}{\partial x \partial t}(x,t) = \frac{\partial^2 e}{\partial x^2}(x,t) \quad \text{and}$$
$$\frac{\partial^2 p}{\partial t^2}(x,t) = \frac{\partial^2 e}{\partial t \partial x}(x,t) = \frac{\partial^2 e}{\partial x \partial t}(x,t) = \frac{\partial^2 p}{\partial x^2}(x,t).$$

7.5 Suppose that u is a solution of the wave equation, so $\partial_2^2 u - c^2 \partial_1^2 u = 0$.

(a) For a fixed $y \in \mathbf{R}$, $\partial_x v(x,t) = \partial_1 u(x-y,t)$, $\partial_{xx} v(x,t) = \partial_1^2 u(x-y,t)$, $\partial_t v(x,t) = \partial_2 u(x-y,t)$ and $\partial_{tt} v(x,t) = \partial_2^2 u(x-y,t)$, Thus

$$\partial_{tt} v(x,t) - c^2 \partial_{xx} v(x,t) = \partial_2^2 u(x-y,t) - c^2 \partial_1^2 u(x-y,t) = (\partial_2^2 u - c^2 \partial_1^2 u)(x-y,t) = 0.$$

(b) For a fixed $a \in \mathbf{R}$, $\partial_x w(x,t) = a\partial_1 u(ax,at)$, $\partial_{xx} w(x,t) = a^2 \partial_1^2 u(ax,at)$, $\partial_t w(x,t) = a\partial_2 u(ax,at)$ and $\partial_{tt} w(x,t) = a^2 \partial_2^2 u(ax,at)$, Thus

$$\partial_{tt} w(x,t) - c^2 \partial_{xx} w(x,t) = a^2 \partial_2^2 u(ax,at) - c^2 a^2 \partial_1^2 u(ax,at) = a^2(\partial_2^2 u - c^2 \partial_1^2 u)(ax,at) = 0.$$

7.6 Consider a solution u to the *damped string equation*

$$\partial_{tt} u(x,t) - c^2 \partial_{xx} u(x,t) + r\partial_t u(x,t) = 0$$

for $c^2 = T/\rho$ and $T, \rho, r > 0$. Define the energy by the same formula we used in class:

$$E[u](t) = \frac{1}{2} \int_{-\infty}^{\infty} \rho(\partial_t u(x,t))^2 + T(\partial_x u(x,t))^2 dx.$$

We have

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} \rho(\partial_t u(x,t))^2 dx\right) &= \frac{1}{2}\int_{-\infty}^{\infty} \rho\partial_t u(x,t)\partial_{tt} u(x,t) dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \partial_t u(x,t)(T\partial_{xx} u(x,t) - r\rho\partial_t u(x,t)) dx \\
&= -\frac{1}{2}\int_{-\infty}^{\infty} T\partial_{tx} u(x,t)\partial_x u(x,t) dx - \frac{1}{2}\int_{-\infty}^{\infty} r\rho(\partial_t u(x,t))^2 dx \\
&= -\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} T(\partial_x u(x,t))^2 dx\right) - \frac{1}{2}\int_{-\infty}^{\infty} r\rho(\partial_t u(x,t))^2 dx.
\end{aligned}$$

Therefore

$$E[u]'(t) = \frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} \rho(\partial_t u(x,t))^2 + T(\partial_x u(x,t))^2 dx\right) = -\frac{1}{2}\int_{-\infty}^{\infty} r\rho(\partial_t u(x,t))^2 dx \le 0,$$

hence the energy $E[u]$ is a non-increasing function.