Partial Differential Equations (TATA27)
Spring Semester 2019
Solutions to Homework 6

6.1 Consider the function $\Phi\colon \mathbf{R}^n \setminus \{\mathbf{0}\} \to \mathbf{R}$ defined by

$$\Phi(\mathbf{x}) = \begin{cases} -\frac{1}{2\alpha(2)} \ln |\mathbf{x}| & \text{if } n = 2, \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|\mathbf{x}|^{n-2}} & \text{if } n > 2, \end{cases}$$

where $\alpha(n)$ is the volume of the unit ball in $\mathbf{R}^n$. (So, in particular, $\alpha(2) = \pi$ and $\alpha(3) = 4\pi/3$.)

(a) For $\mathbf{x} = (x_1, x_2, \ldots, x_n) \neq \mathbf{0}$ we have

$$\frac{\partial \Phi}{\partial x_j}(\mathbf{x}) = \frac{-x_j}{n\alpha(n)} \left(\sum_{k=1}^n x_k^2 \right)^{-n/2}$$

and

$$\frac{\partial^2 \Phi}{\partial x_j^2}(\mathbf{x}) = \frac{-1}{n\alpha(n)} \left(\sum_{k=1}^n x_k^2 \right)^{-n/2} + \frac{x_j^2}{\alpha(n)} \left(\sum_{k=1}^n x_k^2 \right)^{-(n+2)/2}.$$

Therefore

$$\Delta\Phi(\mathbf{x}) = \sum_{j=1}^n \frac{\partial^2 \Phi}{\partial x_j^2}(\mathbf{x}) = \frac{-1}{\alpha(n)} \left(\sum_{k=1}^n x_k^2 \right)^{-n/2} + \frac{\sum_{j=1}^n x_j^2}{\alpha(n)} \left(\sum_{k=1}^n x_k^2 \right)^{-(n+2)/2} = 0.$$

(b) From (a) we see that

$$\nabla\Phi(\mathbf{x}) = \frac{-1}{n\alpha(n)} \frac{\mathbf{x}}{|\mathbf{x}|^n}$$

so

$$\frac{\partial \Phi}{\partial \mathbf{n}}(\mathbf{x}) = \mathbf{n} \cdot \nabla\Phi(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|} \cdot \frac{-1}{n\alpha(n)} \frac{\mathbf{x}}{|\mathbf{x}|^n} = \frac{-1}{n\alpha(n)} \frac{1}{|\mathbf{x}|^{n-1}}.$$

6.2 We want to prove the following.

Lemma. *Let Ω be an open bounded set with C^1 boundary and suppose that $u \in C^2(\overline{\Omega})$ is such that $\Delta u = f$ for some $f \in C(\overline{\Omega})$. Then*

$$u(\mathbf{x}) = \int_{\partial\Omega} \left\{ \Phi(\mathbf{y}-\mathbf{x}) \left(\frac{\partial u}{\partial \mathbf{n}} \right)(\mathbf{y}) - \left(\frac{\partial \Phi}{\partial \mathbf{n}} \right)(\mathbf{y}-\mathbf{x}) u(\mathbf{y}) \right\} d\sigma(\mathbf{y}) - \int_\Omega f(\mathbf{y})\Phi(\mathbf{y}-\mathbf{x}) d\mathbf{y}. \quad (1)$$

for each $\mathbf{x} \in \Omega$.

Proof. We wish to apply Green's second identity (5.10) to the functions $\Phi(\cdot - \mathbf{x})$ and u in Ω. However we cannot as $\Phi(\cdot - \mathbf{x})$ is not defined at $\mathbf{x}$. We instead apply (5.10) to $\Omega_r := \Omega \setminus \overline{B_r(\mathbf{x})}$. We obtain

$$\int_{\Omega_r} f(\mathbf{y})\Phi(\mathbf{y}-\mathbf{x}) d\mathbf{y} = \int_{\partial\Omega_r} \left\{ \Phi(\mathbf{y}-\mathbf{x}) \left(\frac{\partial u}{\partial \mathbf{n}} \right)(\mathbf{y}) - \left(\frac{\partial \Phi}{\partial \mathbf{n}} \right)(\mathbf{y}-\mathbf{x}) u(\mathbf{y}) \right\} d\sigma(\mathbf{y}).$$

However, $\partial\Omega_r$ has two components, $\partial\Omega$ and $\partial B_r(\mathbf{x})$. The integral over $\partial\Omega$ appears on the right-hand side of (1), so we wish to calculate the integral over $\partial B_r(\mathbf{x})$. From (5.11) it is clear that Φ is a radial function—that is, we can write $\Phi(\mathbf{x}) = \phi(|\mathbf{x}|)$ for

$$\phi(r) = \begin{cases} -\frac{1}{2\alpha(2)} \ln |r| & \text{if } n = 2, \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{r^{n-2}} & \text{if } n > 2. \end{cases}$$

Remembering that the outward normal $\mathbf{n}$ to Ω_r is actually an inward normal to $B_r(\mathbf{x})$ on $\partial B_r(\mathbf{x})$, we have from (5.12) that

$$\frac{\partial \Phi}{\partial \mathbf{n}}(\mathbf{y}-\mathbf{x}) = \frac{1}{n\alpha(n)} \frac{1}{r^{n-1}}.$$

Thus

$$\begin{aligned}
&\int_{\partial B_r(\mathbf{x})} \left\{ \Phi(\mathbf{y}-\mathbf{x}) \left(\frac{\partial u}{\partial \mathbf{n}}\right)(\mathbf{y}) - \left(\frac{\partial \Phi}{\partial \mathbf{n}}\right)(\mathbf{y}-\mathbf{x}) u(\mathbf{y}) \right\} d\sigma(\mathbf{y}) \\
=&\int_{\partial B_r(\mathbf{x})} \phi(r) \left(\frac{\partial u}{\partial \mathbf{n}}\right)(\mathbf{y}) d\sigma(\mathbf{y}) - \int_{\partial B_r(\mathbf{x})} \frac{1}{n\alpha(n)} \frac{1}{r^{n-1}} u(\mathbf{y}) d\sigma(\mathbf{y}) \\
=&\phi(r) \left(\int_{\partial B_r(\mathbf{x})} \left(\frac{\partial u}{\partial \mathbf{n}}\right)(\mathbf{y}) d\sigma(\mathbf{y}) \right) - \left(\frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B_r(\mathbf{x})} u(\mathbf{y}) d\sigma(\mathbf{y}) \right).
\end{aligned}$$

Applying the divergence theorem (with $-\mathbf{n}$ being the outward unit normal), we see that

$$\left(\int_{\partial B_r(\mathbf{x})} \left(\frac{\partial u}{\partial \mathbf{n}}\right)(\mathbf{y}) d\sigma(\mathbf{y}) \right) = - \int_{B_r(\mathbf{x})} \Delta u(\mathbf{y}) d\mathbf{y} = - \int_{B_r(\mathbf{x})} f(\mathbf{y}) d\mathbf{y}.$$

Since $f \in C(\overline{\Omega})$ we know it is bounded, say by M, so

$$\phi(r) \left(\int_{\partial B_r(\mathbf{x})} \left(\frac{\partial u}{\partial \mathbf{n}}\right)(\mathbf{y}) d\sigma(\mathbf{y}) \right) = -\phi(r) \int_{B_r(\mathbf{x})} f(\mathbf{y}) d\mathbf{y} \leq \phi(r) \alpha(n) r^n M \to 0$$

as $r \to 0$.

Since $n\alpha(n)r^{n-1}$ is the surface area of $\partial B_r(\mathbf{x})$ and u is continuous,

$$\lim_{r \to 0} \left(\frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B_r(\mathbf{x})} u(\mathbf{y}) d\sigma(\mathbf{y}) \right) = u(\mathbf{x}).$$

Putting these facts together, and taking the limit $r \to 0$ we see the lemma is proved. [It's worth observing that since f is bounded the integral $\int_{\Omega_r} f(\mathbf{y})\Phi(\mathbf{y}-\mathbf{x}) d\mathbf{y}$ converges absolutely to $\int_\Omega f(\mathbf{y})\Phi(\mathbf{y}-\mathbf{x}) d\mathbf{y}$ as $r \to 0$, since the singularity of Φ is absolutely integrable.] □

6.3 We wish to prove the following theorem.

Theorem. *Let $\Omega \subset \mathbf{R}^n$ be an open bounded set with C^2 boundary, and suppose $h \in C^2(\partial\Omega)$ and $f \in C(\overline{\Omega})$. If G is a Green's function for the Laplacian in Ω then the solution of the boundary value problem*

$$\begin{cases} \Delta u = f & \text{in } \Omega, \text{ and} \\ u = h & \text{on } \partial\Omega, \end{cases}$$

is given by

$$u(\mathbf{x}) = - \int_{\partial\Omega} \left(\frac{\partial G(\mathbf{x}, \cdot)}{\partial \mathbf{n}}(\mathbf{y}) \right) h(\mathbf{y}) d\sigma(\mathbf{y}) - \int_\Omega f(\mathbf{y}) G(\mathbf{x}, \mathbf{y}) d\mathbf{y}. \tag{2}$$

where $(\partial G(\mathbf{x}, \cdot)/\partial \mathbf{n})(\mathbf{y}) := \mathbf{n}(\mathbf{y}) \cdot \nabla_\mathbf{y} G(\mathbf{x}, \mathbf{y})$ is the normal derivative of $\mathbf{y} \mapsto G(\mathbf{x}, \mathbf{y})$.

Proof. Green's second identity (5.10) applied to the functions u and $\mathbf{y} \mapsto G(\mathbf{x}, \mathbf{y}) - \Phi(\mathbf{y}-\mathbf{x})$ tells us

$$-\int_\Omega (G(\mathbf{x},\mathbf{z}) - \Phi(\mathbf{z}-\mathbf{x})) f(\mathbf{z}) d\mathbf{z} = \int_{\partial\Omega} u(\mathbf{z}) \frac{\partial (G(\mathbf{x}, \cdot) - \Phi(\cdot - \mathbf{x}))}{\partial \mathbf{n}}(\mathbf{z}) - (G(\mathbf{x},\mathbf{z}) - \Phi(\mathbf{z}-\mathbf{x})) \frac{\partial u}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}).$$

and from (1) we have

$$u(\mathbf{x}) = \int_{\partial\Omega} \left\{ \Phi(\mathbf{z}-\mathbf{x}) \left(\frac{\partial u}{\partial \mathbf{n}}\right)(\mathbf{z}) - \left(\frac{\partial \Phi}{\partial \mathbf{n}}\right)(\mathbf{z}-\mathbf{x}) u(\mathbf{z}) \right\} d\sigma(\mathbf{z}) - \int_\Omega f(\mathbf{z}) \Phi(\mathbf{z}-\mathbf{x}) d\mathbf{z}.$$

Taking the difference of the two equalities and using the fact that $G(\mathbf{x}, \mathbf{z}) = 0$ for $\mathbf{z} \in \partial\Omega$ gives (2). □

6.4 Let Ω be an open bounded set with C^2 boundary.

(a) i. Fix $\mathbf{x} \in \Omega$. Suppose we have two Green's functions G_1 and G_2 for the Laplacian in Ω. By Definition 5.10 we know that both

$$\mathbf{y} \mapsto G_1(\mathbf{x}, \mathbf{y}) - \Phi(\mathbf{y} - \mathbf{x}) \quad \text{and} \quad \mathbf{y} \mapsto G_2(\mathbf{x}, \mathbf{y}) - \Phi(\mathbf{y} - \mathbf{x})$$

have continuous extensions which belongs to $C^2(\overline{\Omega})$ and are harmonic in Ω. Here Φ is the fundamental solution definted by (5.11). Thus the difference of these two functions

$$\mathbf{y} \mapsto (G_1(\mathbf{x}, \mathbf{y}) - \Phi(\mathbf{y} - \mathbf{x})) - (G_2(\mathbf{x}, \mathbf{y}) - \Phi(\mathbf{y} - \mathbf{x})) = G_1(\mathbf{x}, \mathbf{y}) - G_2(\mathbf{x}, \mathbf{y})$$

also has a continuous extension which belongs to $C^2(\overline{\Omega})$ and is harmonic in Ω.

ii. By Definition 5.10 we know that $G_1(\mathbf{x}, \mathbf{y}) = G_2(\mathbf{x}, \mathbf{y}) = 0$ for $\mathbf{y} \in \partial\Omega$. Thus $\mathbf{y} \mapsto G_1(\mathbf{x}, \mathbf{y}) - G_2(\mathbf{x}, \mathbf{y})$ solves (5.2) with $f = 0$ and $g = 0$. Since we know from Section 5.2, there is at most one continuous solution to (5.2) and the zero function is also a solution, we can conclude that $G_1(\mathbf{x}, \mathbf{y}) - G_2(\mathbf{x}, \mathbf{y}) = 0$. Thus there is at most one Green's function for a given Ω.

(b) i. Fix $\mathbf{x}, \mathbf{y} \in \Omega$ with $\mathbf{x} \neq \mathbf{y}$ and consider the functions $\mathbf{z} \mapsto u(\mathbf{z}) := G(\mathbf{x}, \mathbf{z})$ and $\mathbf{z} \mapsto v(\mathbf{z}) := G(\mathbf{y}, \mathbf{z})$. We apply Green's second identity (5.10) to u and v in the domain $\Omega_r := \Omega \setminus \overline{(B_r(\mathbf{x}) \cup B_r(\mathbf{y}))}$ for $r > 0$ so small that $\overline{(B_r(\mathbf{x}) \cup B_r(\mathbf{y}))} \subset \Omega$ and $\overline{B_r(\mathbf{x})} \cap \overline{B_r(\mathbf{y})} = \emptyset$. We obtain

$$\begin{aligned}
0 &= \int_{\partial\Omega_r} G(\mathbf{x}, \mathbf{z}) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) - G(\mathbf{y}, \mathbf{z}) \frac{\partial G(\mathbf{x}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) \\
&= \int_{\partial\Omega} G(\mathbf{x}, \mathbf{z}) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) - G(\mathbf{y}, \mathbf{z}) \frac{\partial G(\mathbf{x}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) \\
&\quad + \int_{\partial B_r(\mathbf{x})^c} G(\mathbf{x}, \mathbf{z}) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) - G(\mathbf{y}, \mathbf{z}) \frac{\partial G(\mathbf{x}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) \\
&\quad + \int_{\partial B_r(\mathbf{y})^c} G(\mathbf{x}, \mathbf{z}) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) - G(\mathbf{y}, \mathbf{z}) \frac{\partial G(\mathbf{x}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) \\
&= - \int_{\partial B_r(\mathbf{x})} G(\mathbf{x}, \mathbf{z}) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) - G(\mathbf{y}, \mathbf{z}) \frac{\partial G(\mathbf{x}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) \\
&\quad - \int_{\partial B_r(\mathbf{y})} G(\mathbf{x}, \mathbf{z}) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) - G(\mathbf{y}, \mathbf{z}) \frac{\partial G(\mathbf{x}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}).
\end{aligned}$$

since $G(\mathbf{x}, \mathbf{z}) = G(\mathbf{y}, \mathbf{z}) = 0$ for $z \in \partial\Omega$.

ii. Since we know that $\mathbf{z} \mapsto G(\mathbf{x}, \mathbf{z}) - \Phi(\mathbf{z} - \mathbf{x})$ has a $C^2(\overline{\Omega})$ extension, we know that $\mathbf{z} \mapsto G(\mathbf{x}, \mathbf{z}) - \Phi(\mathbf{z} - \mathbf{x})$ is bounded near $\mathbf{x}$. We also know that $\partial G(\mathbf{y}, \cdot)/\partial \mathbf{n}$ is bounded near $\mathbf{x}$, since $\mathbf{x} \neq \mathbf{y}$. This means that the integrand in

$$\int_{\partial B_r(\mathbf{x})} (G(\mathbf{x}, \mathbf{z}) - \Phi(\mathbf{z} - \mathbf{x})) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z})$$

is bounded. This fact together with the fact that the measure of $\partial B_r(\mathbf{x})$ tends to zero as $r \to 0$, means the integral above tends to zero as $r \to 0$. By the same logic

$$\int_{\partial B_r(\mathbf{x})} G(\mathbf{y}, \mathbf{z}) \frac{\partial (G(\mathbf{x}, \cdot) - \Phi(\cdot - \mathbf{x}))}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) \to 0$$

as $r \to 0$.

iii. Since Φ is a radial function, we can write $\Phi(\mathbf{z} - \mathbf{x}) = \phi(r)$ when $\mathbf{z} \in \partial B_r(\mathbf{x})$. This fact and the divergence theorem give

$$\begin{aligned}
&\int_{\partial B_r(\mathbf{x})} \Phi(\mathbf{z} - \mathbf{x}) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) \\
&= \phi(r) \int_{\partial B_r(\mathbf{x})} \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) \\
&= \phi(r) \int_{\partial B_r(\mathbf{x})} \Delta(G(\mathbf{y}, \cdot))(\mathbf{z}) d\sigma(\mathbf{z}) = 0.
\end{aligned}$$

By (5.12) (homework question 6.1) we can compute

$$\begin{aligned}
&\int_{\partial B_r(\mathbf{x})} G(\mathbf{y},\mathbf{z})\frac{\partial \Phi}{\partial \mathbf{n}}(\mathbf{z}-\mathbf{x})d\sigma(\mathbf{z})\\
&= \int_{\partial B_r(\mathbf{x})} G(\mathbf{y},\mathbf{z})\frac{-1}{n\alpha(n)}\frac{1}{|\mathbf{z}-\mathbf{x}|^{n-1}}d\sigma(\mathbf{z})\\
&= \frac{-1}{n\alpha(n)}\frac{1}{r^{n-1}}\int_{\partial B_r(\mathbf{x})} G(\mathbf{y},\mathbf{z})d\sigma(\mathbf{z})\\
&= -G(\mathbf{y},\mathbf{x})
\end{aligned}$$

by the mean value property for harmonic functions (Theorem 5.6).

iv. We can compute

$$\begin{aligned}
&\int_{\partial B_r(\mathbf{x})} G(\mathbf{x},\mathbf{z})\frac{\partial G(\mathbf{y},\cdot)}{\partial \mathbf{n}}(\mathbf{z}) - G(\mathbf{y},\mathbf{z})\frac{\partial G(\mathbf{x},\cdot)}{\partial \mathbf{n}}(\mathbf{z})d\sigma(\mathbf{z})\\
&= \int_{\partial B_r(\mathbf{x})} (G(\mathbf{x},\mathbf{z})-\Phi(\mathbf{z}-\mathbf{x}))\frac{\partial G(\mathbf{y},\cdot)}{\partial \mathbf{n}}(\mathbf{z})d\sigma(\mathbf{z}) + \int_{\partial B_r(\mathbf{x})} \Phi(\mathbf{z}-\mathbf{x})\frac{\partial G(\mathbf{y},\cdot)}{\partial \mathbf{n}}(\mathbf{z})d\sigma(\mathbf{z})\\
&\quad - \int_{\partial B_r(\mathbf{x})} G(\mathbf{y},\mathbf{z})\left(\frac{\partial G(\mathbf{x},\cdot)}{\partial \mathbf{n}}(\mathbf{z}) - \frac{\partial \Phi}{\partial \mathbf{n}}(\mathbf{z}-\mathbf{x})\right)d\sigma(\mathbf{z}) - \int_{\partial B_r(\mathbf{x})} G(\mathbf{y},\mathbf{z})\frac{\partial \Phi}{\partial \mathbf{n}}(\mathbf{z}-\mathbf{x})d\sigma(\mathbf{z})\\
&\to 0+0-0+G(\mathbf{y},\mathbf{x})
\end{aligned}$$

as $r \to 0$ by our work above.

v. The statement we obtain is

$$\int_{\partial B_r(\mathbf{y})} G(\mathbf{y},\mathbf{z})\frac{\partial G(\mathbf{x},\cdot)}{\partial \mathbf{n}}(\mathbf{z}) - G(\mathbf{x},\mathbf{z})\frac{\partial G(\mathbf{y},\cdot)}{\partial \mathbf{n}}(\mathbf{z})d\sigma(\mathbf{z}) \to G(\mathbf{x},\mathbf{y}).$$

which is the negative of the second term on the right-hand side of (†). Thus, in the limit $r \to 0$, (†) becomes $0 = G(\mathbf{y},\mathbf{x}) - G(\mathbf{x},\mathbf{y})$.