**Partial Differential Equations (TATA27)**
**Spring Semester 2019**
Solutions to Homework 5

5.1 Suppose $u \in C^2(\overline{\Omega})$ satisfies $\Delta u - \lambda u = 0$ in $\Omega$ and $v \in C^1(\overline{\Omega})$ is such that $v(\mathbf{x}) = u(\mathbf{x})$ for all $\mathbf{x} \in \partial\Omega$. Then, setting $w = v - u$ and using Green's first identity (5.8), we see that

$$\begin{aligned}
E_\lambda[v] &= \frac{1}{2}\int_\Omega (|\nabla v(\mathbf{x})|^2 + \lambda|v(\mathbf{x})|^2)d\mathbf{x} \\
&= \frac{1}{2}\int_\Omega (|\nabla w(\mathbf{x}) + \nabla u(\mathbf{x})|^2 + \lambda|w(\mathbf{x}) + u(\mathbf{x})|^2)d\mathbf{x} \\
&= \frac{1}{2}\int_\Omega (|\nabla w(\mathbf{x})|^2 + \lambda|w(\mathbf{x})|^2)d\mathbf{x} + E_\lambda[u] + \int_\Omega (\nabla w(\mathbf{x}) \cdot \nabla u(\mathbf{x}) + \lambda w(\mathbf{x})u(\mathbf{x}))d\mathbf{x} \\
&= \frac{1}{2}\int_\Omega (|\nabla w(\mathbf{x})|^2 + \lambda|w(\mathbf{x})|^2)d\mathbf{x} + E_\lambda[u] + \int_\Omega w(\mathbf{x})(-\Delta u(\mathbf{x}) + \lambda u(\mathbf{x}))d\mathbf{x} \\
&= \frac{1}{2}\int_\Omega (|\nabla w(\mathbf{x})|^2 + \lambda|w(\mathbf{x})|^2)d\mathbf{x} + E_\lambda[u] \\
&\geq E_\lambda[u].
\end{aligned}$$

5.2 Suppose $u \in C^2(\overline{\Omega})$ satisfies the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \text{ and} \\ \frac{\partial u}{\partial \mathbf{n}} := \mathbf{n} \cdot \nabla u = h & \text{on } \partial\Omega \end{cases}$$

For $v \in C^1(\overline{\Omega})$ set $w = v - u$. Then, using Green's first identity (5.8),

$$\begin{aligned}
E_h[v] &= \frac{1}{2}\int_\Omega |\nabla v(\mathbf{x})|^2 d\mathbf{x} - \int_{\partial\Omega} h(\mathbf{x})v(\mathbf{x})d\sigma(\mathbf{x}) \\
&= \frac{1}{2}\int_\Omega |\nabla w(\mathbf{x}) + \nabla u(\mathbf{x})|^2 d\mathbf{x} - \int_{\partial\Omega} h(\mathbf{x})(w(\mathbf{x}) + u(\mathbf{x}))d\sigma(\mathbf{x}) \\
&= \frac{1}{2}\int_\Omega |\nabla w(\mathbf{x})|^2 d\mathbf{x} + E_h[u] + \int_\Omega \nabla w(\mathbf{x}) \cdot \nabla u(\mathbf{x})d\mathbf{x} - \int_{\partial\Omega} h(\mathbf{x})w(\mathbf{x})d\sigma(\mathbf{x}) \\
&= \frac{1}{2}\int_\Omega |\nabla w(\mathbf{x})|^2 d\mathbf{x} + E_h[u] - \int_\Omega w(\mathbf{x})\Delta u(\mathbf{x})d\mathbf{x} + \int_{\partial\Omega} \left(\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) - h(\mathbf{x})\right) w(\mathbf{x})d\sigma(\mathbf{x}) \\
&= \frac{1}{2}\int_\Omega |\nabla w(\mathbf{x})|^2 d\mathbf{x} + E_h[u] \\
&\geq E_h[u].
\end{aligned}$$