Partial Differential Equations (TATA27)
Spring Semester 2019
Solutions to Homework 3

3.1 Consider two points $\mathbf{x}, \mathbf{y} \in \mathbf{R}^2$ with polar coordinates (r, θ) and (a, ϕ), respectively. Looking at Figure 1 we can see that the right-angled triangle with vertices $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ has hypotenuse of length $|\mathbf{x} - \mathbf{y}|$ and the other two sides are of length $r - a\cos(\phi - \theta)$ and $a\sin(\phi - \theta)$. Thus, by Pythagoras' theorem

$$\begin{aligned} |\mathbf{x} - \mathbf{y}|^2 &= (r - a\cos(\phi - \theta))^2 + (a\sin(\phi - \theta))^2 \\ &= r^2 - 2ar\cos(\phi - \theta) + a^2(\cos^2(\phi - \theta) + \sin^2(\phi - \theta)) \\ &= r^2 - 2ar\cos(\phi - \theta) + a^2. \end{aligned} \tag{1}$$

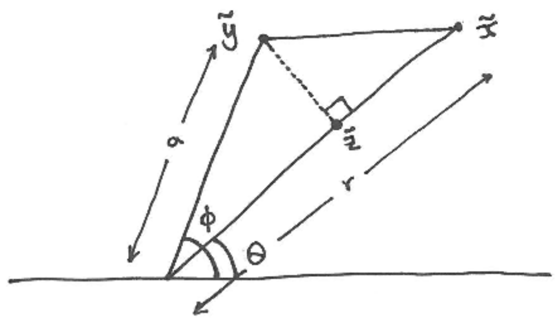

Figure 1: Two points $\mathbf{x}, \mathbf{y} \in \mathbf{R}^2$ with polar coordinates (r, θ) and (a, ϕ), respectively.

A line integral of $f\colon \gamma \to \mathbf{R}$ over a curve $\gamma \subset \mathbf{R}^2$ is defined to be

$$\int_\gamma f(\mathbf{y}) d\sigma(\mathbf{y}) = \int_0^{2\pi} f(\mathbf{r}(\phi))|\mathbf{r}'(\phi)| d\phi$$

where $\mathbf{r}\colon [0, 2\pi] \to \mathbf{R}$ is a parametrisation of γ. Thus, if we take the parametrisation $\mathbf{r}(\phi) = (a\cos\phi, a\sin\phi)$ of the cicle $\{\mathbf{y} \in \mathbf{R}^2 \mid |\mathbf{y}| = a\}$, then $|\mathbf{r}'(\phi)| = a$ and

$$\int_{|\mathbf{y}|=a} \frac{\tilde{h}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2} d\sigma(\mathbf{y}) = \int_0^{2\pi} \frac{\tilde{h}(\mathbf{r}(\phi))}{|\mathbf{x} - \mathbf{r}(\phi)|^2} a d\phi = a \int_0^{2\pi} \frac{h(\phi)}{r^2 - 2ar\cos(\phi - \theta) + a^2} d\phi,$$

where we used (1) (observing $\mathbf{r}(\phi)$ has polar coordinates (a, ϕ)). Since $|\mathbf{x}|^2 = r^2$, this implies

$$\frac{(a^2 - |\mathbf{x}|^2)}{2\pi a} \int_{|\mathbf{y}|=a} \frac{\tilde{h}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2} d\sigma(\mathbf{y}) = \frac{(a^2 - r^2)}{2\pi} \int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2} d\phi.$$

3.2 We search for solutions of the form $u(r, \theta) = R(r)\Theta(\theta)$ via the method of separation of variables, just as we did in Section 5.3.2. Exactly as before we wish to solve the two separate ODEs

$$\Theta'' + \lambda\Theta = 0 \quad \text{and} \quad r^2 R'' + rR' - \lambda R = 0,$$

for $\lambda \in \mathbf{R}$, but instead of wanting to find periodic Θ as we did for the disc, the boundary conditions $u(r, 0) = u(r, \beta) = 0$ imply we need

$$\Theta(0) = \Theta(\beta) = 0.$$

Solving the ODE for Θ with these boundary conditions gives

$$\lambda = \left(\frac{m\pi}{\beta}\right)^2 \quad \text{and} \quad \Theta(\theta) = \sin(m\pi\theta/\beta)$$

for $m = 1, 2, \ldots$. We now solve the ODE for R, which is of Euler form. We find that $R(r) = r^\alpha$, where $\alpha^2 = \lambda$. We reject negative exponents α, as they produce solutions R which are not continuous at the origin (the vertex of the wedge).[1] Thus we have a solution

$$u(r,\theta) = R(r)\Theta(\theta) = r^{m\pi/\beta}\sin(m\pi\theta/\beta)$$

for each positive integer m. In order to try to satisfy the boundary condition $u(a,\theta) = h(\theta)$ we consider linear combinations of these,

$$u(r,\theta) = \sum_{m=1}^{\infty} A_m r^{m\pi/\beta}\sin(m\pi\theta/\beta),$$

and consider the boundary value

$$h(\theta) = u(a,\theta) = \sum_{m=1}^{\infty} A_m a^{m\pi/\beta}\sin(m\pi\theta/\beta),$$

which has the form of a Fourier sine series for h, so it is natural to choose

$$A_m = \frac{2}{\beta a^{m\pi/\beta}}\int_0^\beta h(\phi)\sin(m\pi\phi/\beta)d\phi$$

and so

$$u(r,\theta) = \frac{2}{\beta}\sum_{m=1}^{\infty}\left(\frac{r}{a}\right)^{m\pi/\beta}\int_0^\beta h(\phi)\sin(m\pi\phi/\beta)\sin(m\pi\theta/\beta)d\phi.$$

3.3 We search for a solution which separates in the Cartesian coordinates x and y, that is, we search for a solution u of the form $u(x,y) = X(x)Y(y)$. In this case the equation $\Delta u(x,y) = 0$ can be rewritten as $X''(x)Y(y) + X(x)Y''(y) = 0$, and will be fullfiled if X and Y satisfy

$$X''(x) = \lambda X(x) \quad \text{and} \quad Y''(x) = -\lambda Y(x)$$

for some constant λ.

For negative λ, the general solution for Y has the form

$$Y(y) = Ae^{\sqrt{-\lambda}y} + Be^{-\sqrt{-\lambda}y}.$$

But the only choice of constants A and B which can satisfy the first two boundary conditions $Y'(0) = Y'(\pi) = 0$ is $A = B = 0$.

For non-negative λ, the general solutions for Y has the form

$$Y(y) = Ae^{i\sqrt{\lambda}y} + Be^{-i\sqrt{\lambda}y},$$

when $\lambda > 0$ or $Y(y) = A + By$ in the case $\lambda = 0$. The boundary conditions $Y'(0) = Y'(\pi) = 0$ require $A = B$ and $\lambda = n^2$ for $n = 1, 2, \ldots$ and $B = 0$ when $\lambda = 0$. Thus for each non-negative integer n,

$$Y_n(y) = A_n\cos(ny)$$

solves $Y_n''(x) = -\lambda_n Y_n(x)$ with $\lambda_n = n^2$.

Functions X_n which solve $X_n''(x) = \lambda_n X_n(x)$ and the boundary condition $X_n(0) = 0$ are $X_n(x) = \sinh(nx)$ for positive n and $X_0(x) = x$.

Since the Laplacian is a linear operator and the first three boundary conditions are homogeneous, we can take linear combinations of $X_n(x)Y_n(y)$ to constructed harmonic functions which satisfies the first three boundary conditions:

$$u(x,y) = A_0 x + \sum_{n=1}^{\infty} A_n\sinh(nx)\cos(ny).$$

[1] It is of course interesting to think about what would happen if we do not impose such continuity, but makes the solution more involved.

In order to choose A_n so that the last boundary condition is satisfied, we write

$$\cos^2(y) = \frac{\cos(2y)+1}{2}$$

and so we require

$$\frac{\cos(2y)+1}{2} = \cos^2(y) = u(\pi, y) = A_0\pi + \sum_{n=1}^{\infty} A_n \sinh(n\pi)\cos(ny).$$

This can be achieved by choosing $A_0 = 1/(2\pi)$, $A_2 = 1/(2\sinh(2\pi))$ and all the other A_n equal to zero. Thus, our sought-after function is

$$u(x, y) = \frac{x}{2\pi} + \frac{\sinh(2x)\cos(2y)}{2\sinh(2\pi)}.$$