**Partial Differential Equations (TATA27)**
**Spring Semester 2019**
Solutions to Homework 1

1.1 (a) For $u, v\colon \mathbf{R}^n \to \mathbf{R}$ and $a, b \in \mathbf{R}$, we have

$$\begin{aligned}\nabla(au+bv) &= (\partial_1(au+bv), \partial_2(au+bv), \dots, \partial_n(au+bv))\\ &= a(\partial_1 u, \partial_2 u, \dots, \partial_n u) + b(\partial_1 v, \partial_2 v, \dots, \partial_n v) = a\nabla u + b\nabla v,\end{aligned}$$

so $\nabla$ is linear.

(b) For $u = (u^1, u^2, \dots, u^n)$ and $v = (v^1, v^2, \dots, v^n)$, functions from $\mathbf{R}^n$ to $\mathbf{R}^n$, and $a, b \in \mathbf{R}$, we have

$$\begin{aligned}\operatorname{div}(au+bv) &= \sum_{j=1}^n \partial_j(au^j + bv^j) = \sum_{j=1}^n (a\partial_j u^j + b\partial_j v^j)\\ &= a\left(\sum_{j=1}^n \partial_j u^j\right) + b\left(\sum_{j=1}^n \partial_j v^j\right) = a \operatorname{div} u + b \operatorname{div} v,\end{aligned}$$

so div is linear.

(c) For $u = (u^1, u^2, u^3)$ and $u = (u^1, u^2, u^3)$, functions from $\mathbf{R}^3$ to $\mathbf{R}^3$, and $a, b \in \mathbf{R}$, we have

$$\begin{aligned}&\operatorname{curl}(au+bv)\\ &= \big(\partial_2(au_3+bv_3) - \partial_3(au_2+bv_2), \partial_3(au_1+bv_1) - \partial_1(au_3+bv_3), \partial_1(au_2+bv_2) - \partial_2(au_1+bv_1)\big)\\ &= \big(a\partial_2 u_3 + b\partial_2 v_3 - a\partial_3 u_2 - b\partial_3 v_2, a\partial_3 u_1 + b\partial_3 v_1 - a\partial_1 u_3 - b\partial_1 v_3, a\partial_1 u_2 + b\partial_1 v_2 - a\partial_2 u_1 - b\partial_2 v_1\big)\\ &= a\big(\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1\big) + b\big(\partial_2 v_3 - \partial_3 v_2, \partial_3 v_1 - \partial_1 v_3, \partial_1 v_2 - \partial_2 v_1\big)\\ &= a \operatorname{curl} u + b \operatorname{curl} v,\end{aligned}$$

so curl is linear.

(d) For $u, v\colon \mathbf{R}^n \to \mathbf{R}$ and $a, b \in \mathbf{R}$, we have

$$\begin{aligned}\Delta(au+bv) &= \sum_{j=1}^n \partial_j^2(au+bv) = \sum_{j=1}^n \partial_j(a\partial_j u + b\partial_j v)\\ &= \sum_{j=1}^n (a\partial_j^2 u + b\partial_j^2 v) = a\left(\sum_{j=1}^n \partial_j^2 u\right) + b\left(\sum_{j=1}^n \partial_j^2 v\right)\\ &= a\Delta u + b\Delta v,\end{aligned}$$

so $\Delta$ is linear.

1.2 We can see directly that the orders of the equations are (a) 2, (b) 2, (c) 4 and (d) 1. It is also easy to check that (a) and (d) are linear—which we do by check that if $u$ and $v$ are solutions, and $a$ and $b$ are constants, then $au + bv$ is a solution.

Equation (b) is also linear, although non-homogeneous. Indeed we can write the equation as $\mathcal{L}u = -x^2$ where $\mathcal{L} = \partial_t^2 - \partial_x^2$. We can check that $\mathcal{L}$ is a linear operator by considering two functions $u$ and $v$ and two constants $\alpha$ and $\beta$, and calculating

$$\mathcal{L}(\alpha u + \beta v) = (\alpha u + \beta v u + \beta v)_{tt} - (\alpha u + \beta v)_{xx} = \alpha u_{tt} - \alpha u_{xx} + \beta v_{tt} - \beta v_{xx} = \alpha\mathcal{L}(u) + \beta\mathcal{L}(v).$$

We can see that (c) is non-linear, as although the first two terms in the equation are linear, the third is not. Indeed $\sqrt{1+u} + \sqrt{1+v} = \sqrt{1+(u+v)}$ only if $4(1+u)(1+v) = 1$, so $u \mapsto u_t + u_{xxxx} + \sqrt{1+u}$ is not a linear operator.

1.3 We look for curves $t \mapsto (X(t), Y(t))$ on which a solution to

$$u_x(x,y) + yu_y(x,y) = 0 \quad \text{for all } x, y \in \mathbf{R}$$

will be constant. Thus, setting $z(t) = u(X(t), Y(t))$, we require

$$0 = z'(t) = X'(t)u_x(X(t), Y(t)) + Y'(t)u_y(X(t), Y(t))$$

so, comparing this equality with the PDE, we choose

$$X'(t) = 1 \quad \text{and}$$
$$Y'(t) = Y(t).$$

Thus, $X(t) = t + c_1$ and $Y(t) = c_2 e^t$. Let's now consider a characteristic curve which passes through an arbitrary point $(x, y)$, say when $t = 0$. Such a curve will satisfy $x = X(0) = 0 + c_1$ and $y = Y(0) = c_2 e^0$, so $c_1 = x$ and $c_2 = y$ and we have $X(t) = t + x$ and $Y(t) = ye^t$.

To calculate the value of $u(x, y)$ we will use the fact $u$ is constant along the characteristic curves and the fact we know $u(0, y) = g(y)$. We need to calculate for which $t$ our characteristic curve will cross the $y$-axis. This happens when $0 = X(t) = t + x$, that is when $t = -x$. $Y(-x) = ye^{-x}$, so we have $u(x, t) = u(0, ye^{-x}) = g(ye^{-x})$. Thus we have calculated that the solution must be

$$u(x, y) = g(ye^{-x}).$$

We can check directly that this is indeed a solution.

1.4 We wish to find curves $t \mapsto (X(t), Y(t))$ on which a solution to

$$(1 + x^2)u_x(x, y) + u_y(x, y) = 0 \quad \text{for all } x, y \in \mathbf{R}$$

will be constant. Thus, setting $z(t) = u(X(t), Y(t))$, we require

$$0 = z'(t) = X'(t)u_x(X(t), Y(t)) + Y'(t)u_y(X(t), Y(t))$$

so, comparing this equality with the PDE, we choose

$$X'(t) = 1 + X(t)^2 \quad \text{and}$$
$$Y'(t) = 1.$$

Thus, $Y(t) = t + c_Y$ and

$$t = \int dt = \int \frac{X'(t)}{1 + X(t)^2} dt = \arctan(X(t)) + c_X$$

so $X(t) = \tan(t - c_X)$, for constants $c_X, c_Y \in \mathbf{R}$.

For a given point $(x, y)$, we can find a characteristic curve which passes through $(x, y)$, when say $t = 0$, by taking $Y(t) = t + y$ and $X(t) = \tan(t + \arctan(x))$ (that is, by choosing $c_Y = y$ and $c_X = -\arctan(x)$). Then, using the fact $u$ is constant on characteristic curves and the condition $u(0, y) = g(y)$,

$$\begin{aligned} u(x, y) &= u(X(0), Y(0)) = u(X(-\arctan x), Y(-\arctan x)) \\ &= u(-\arctan x + y, 0) \\ &= g(y - \arctan x). \end{aligned}$$

1.5 For fixed $(x, t)$, $z(s) = u(x + bs, t + s)$ so we can compute

$$z'(s) = \frac{d}{ds} u(x + bs, t + s) = b \cdot \nabla u(x + bs, t + s) + u_t(x + bs, t + s),$$

so, using the PDE, we see that

$$z'(s) = f(x + bs, t + s).$$

Integrating with respect to $s$ from $-t$ to 0, we obtain

$$\int_{-t}^{0} f(x + bs, t + s) ds = \int_{-t}^{0} z'(s) ds = z(0) - z(-t) = u(x, t) - u(x - bt, 0) = u(x, t) - g(x - bt),$$

so

$$u(x, t) = g(x - bt) + \int_0^t f(x - bs, t - s) ds. \tag{‡}$$

1.6 We search for appropriate curves $(X, T)$ such that the solution on the curves $s \mapsto z(s) := u(X(s), T(s))$ behaves nicely. We have

$$z'(s) = \frac{d}{ds}u(X(s), T(s)) = X'(s)\partial_1 u(X(s), T(s)) + T'(s)\partial_2 u(X(s), T(s)),$$

so it seems reasonable to set $X'(s) = a$ and $T'(s) = b$. Thus $X(s) = as + c_X$ and $T(s) = bs + c_T$ for constants $c_X, c_T \in \mathbf{R}$. We can then rewrite the PDE as

$$z'(s) + cz(s) = a\partial_1 u(X(s), T(s)) + b\partial_2 u(X(s), T(s)) + cu(X(s), T(s)) = 0.$$

This is an ODE with general solution $z(s) = Ae^{-cs}$ for any $A \in \mathbf{R}$.

Now fix $(x, t)$. If we choose $c_X = x$ and $c_T = t$, then $X(s) = as + x$ and $T(s) = bs + t$, and when $s = 0$ the characteristic curve passes through $(X(0), T(0)) = (x, t)$ and when $s = -t/b$ the curve passes through $(X(-t/b), T(-t/b)) = (x - at/b, 0)$. When $s = -t/b$ we can use the initial condition to find the value of $z$:

$$z(-t/b) = u(X(-t/b), T(-t/b)) = u(x - at/b, 0) = g(x - at/b).$$

But on the other hand, using the form of the general solution to the characteristic ODE, $z(-t/b) = Ae^{ct/b}$, so $A = g(x - at/b)e^{-ct/b}$. Equally, for $s = 0$,

$$u(x, t) = z(0) = Ae^{-c0} = g(x - at/b)e^{-ct/b},$$

which gives us an expression for the solution $u$.