Partial Differential Equations (TATA27)
Spring Semester 2019
Homework 8

Review of previous seminars

In seminars 10 and 11 we covered section 6.5. After checking through the notes to see that you are familiar with the material, try the following exercises.

8.1 Use (6.14) to find a solution to the wave equation (6.10) in three dimensions with initial conditions given by:

(a) $\phi(x, y, z) = 0$ and $\psi(x, y, z) = 1$ for all $(x, y, z) \in \mathbf{R}^3$; and

(b) $\phi(x, y, z) = 0$ and $\psi(x, y, z) = y$ for all $(x, y, z) \in \mathbf{R}^3$.

8.2 Use (6.16) to find a solution to the wave equation (6.15) in two dimensions with initial conditions given by $\phi(x, y) = 0$ and $\psi(x, y) = A$ for all $(x, y) \in \mathbf{R}^2$ and some constant $A \in \mathbf{R}$.

8.3 Given a function $u\colon \mathbf{R}^3 \to \mathbf{R}$ we can define a new function $\overline{u}\colon \mathbf{R}^3 \to \mathbf{R}$ via the formula

$$\overline{u}(\mathbf{x}) = \frac{1}{4\pi r^2} \int_{|\mathbf{y}|=r} u(\mathbf{y}) d\sigma(\mathbf{y})$$

where $r = |\mathbf{x}|$. (The function $\overline{u}$ is said to be *radial* because $\overline{u}(\mathbf{x})$ depends only on $|\mathbf{x}|$.) Prove that $\Delta\overline{u} = \overline{\Delta u}$. [Hint: Here it is easier to compute Δ using spherical polar coordinates.]

8.4 Show that formula (6.8) can be rewritten as

$$v(x, t) = \frac{\partial}{\partial t}\left(\frac{1}{2c}\int_{ct-x}^{ct+x} g_{\text{odd}}(y)dy\right) + \frac{1}{2c}\int_{ct-x}^{ct+x} h_{\text{odd}}(y)dy$$

when $0 \le x \le ct$.

Group work

8.5 A solution $u\colon \mathbf{R}^3 \times [0, \infty) \to \mathbf{R}$ to the wave equation (6.10) in three dimensions is called spherical if $u(\mathbf{x}, t) = u_0(|\mathbf{x}|, t)$ for some function $u_0\colon \mathbf{R} \times [0, \infty) \to \mathbf{R}$ (that is, if it is radial in its spatial variables). This question investigates what form spherical solutions to the wave equation must take.

(a) By arguing similarly to how we proved d'Alembert's formula (6.3) show that an arbitrary solution to the wave equation

$$\partial_t^2 v(r, t) - \partial_r^2 v(r, t) = 0 \quad \text{for } r \in \mathbf{R} \text{ and } t > 0$$

in one dimension has the form

$$v(r, t) = f(x - t) + g(x + t)$$

for some functions $f\colon \mathbf{R} \to \mathbf{R}$ and $g\colon \mathbf{R} \to \mathbf{R}$.

(b) By making use of (6.12), show that spherical solutions of the wave equation (6.10) have the form

$$u(\mathbf{x}, t) = \frac{f(|\mathbf{x}| - t) + g(|\mathbf{x}| + t)}{|\mathbf{x}|}. \tag{$*$}$$

(c) Comment on the smoothness of $(*)$ in relation to the smoothness of f and g. Is there any difference between the one and three dimensional cases?