Partial Differential Equations (TATA27)
Spring Semester 2019
Homework 3

Review of previous seminar

To finish off our work from Seminar 3 read Section 5.2.

Preparation for the next seminar

In preparation for Seminar 4 read through Section 5.3 and attempt the following problem.

3.1 Consider two points $\mathbf{x}, \mathbf{y} \in \mathbf{R}^2$ with polar coordinates (r, θ) and (a, ϕ), respectively. Using a geometric argument (or otherwise) show that

$$|\mathbf{x} - \mathbf{y}|^2 = r^2 - 2ar\cos(\theta - \phi) + a^2.$$

Use this fact to help you rewrite the Poisson formula

$$u(r, \theta) = \frac{(a^2 - r^2)}{2\pi} \int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2} d\phi \tag{5.6}$$

as

$$u(\mathbf{x}) = \frac{(a^2 - |\mathbf{x}|^2)}{2\pi a} \int_{|\mathbf{y}|=a} \frac{\tilde{h}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2} d\sigma(\mathbf{y}). \tag{5.7}$$

Group work

We will work on the following exercise at the end of the seminar then we will discuss possible solutions together in Seminar 5.

3.2 Let $W = \{\mathbf{x} = (r, \theta) \in \mathbf{R}^2 \,|\, 0 < r < a \text{ and } 0 < \theta < \beta\}$ denote a wedge of length a and angle β (where (r, θ) are polar coordinates). Using the same procedure as we used to derive the Poisson formula for D derive a analogous formula for the solution u to

$$\begin{cases} \Delta u = 0 & \text{in } W, \\ u(r, 0) = u(r, \beta) = 0 & \text{for } r \in (0, a), \text{ and} \\ u(a, \theta) = h(\theta) & \text{for } \theta \in (0, \beta). \end{cases}$$

Review exercises

Heres an additional exercise for you to try.

3.3 Using the method of separation of variables find a function $u \colon \overline{S} \to \mathbf{R}$ which is harmonic on the square $S = \{(x, y) \,|\, 0 < x < \pi, 0 < y < \pi\}$ and which satisfies the boundary conditions

$$u_y(x, 0) = u_y(x, \pi) = 0 \quad \text{for } 0 < x < \pi,$$
$$u(0, y) = 0 \quad \text{for } 0 < y < \pi, \text{ and}$$
$$u(\pi, y) = \cos^2 y \quad \text{for } 0 < y < \pi.$$

[Hint: The coordinate system you separate variables in should be chosen based on the geometry of S.]