TATA27 **25th October 2017**

Partial Differential Equations

Comment: *All definitions and theorems have been given in lectures. The remaining content is similar to given homework questions or is work done in lectures.*

1. One day Baby Elwin and his brother Arthur go on a trip to the science museum *Fenomenmagasinet* with their father. They play with a model the Göta Canal and Arthur suggests the following first order PDE as a model for shallow water waves along a thin canal:

$$u_t(x,t) + u(x,t)u_x(x,t) = 0 \quad \text{for all } x \in \mathbf{R} \text{ and } t > 0.$$

Here $u(x,t)$ is the height of the water at time t and a distance x along the canal.

Although the equation is nonlinear the method of characteristics can still be used to construct solutions.

(a) Show that this method leads to the system of ODEs

$$\begin{cases} X'(s) = z(s) \\ T'(s) = 1 \\ z'(s) = 0 \end{cases}$$

where $s \mapsto (X(s), T(s))$ is a characteristic curve in the (x,t)-plane and $z(s) = u(X(s), T(s))$. **[4 marks]**

(b) Motivate why the characteristic curves must be straight lines. **[6 marks]**

(c) Sketch some typical characteristic curves $s \mapsto (X(s), T(s))$ when the solution u satisfies the initial condition

$$u(x,0) = \begin{cases} 0 & \text{if } x < 0; \\ x & \text{if } x \in [0,1]; \\ 1 & \text{if } x > 1. \end{cases}$$

[6 marks]

Solution:

(a) Differentiating $z(s) = u(X(s), T(s))$ we obtain

$$z'(s) = T'(s)u_t(X(s), T(s)) + X'(s)u_x(X(s), T(s)).$$

Comparing the right-hand side with the PDE leads us to choose $T'(s) = 1$ and $X'(s) = u(X(s), T(s)) = z(s)$. With this choice, if u solves the PDE, then

$$z'(s) = u_t(X(s), T(s)) + u(X(s), T(s))u_x(X(s), T(s)) = 0.$$

(b) The equation $z'(s) = 0$ implies z is a constant function, say $z(s) = c$ for some $c \in \mathbf{R}$. Thus $X'(s) = c$ so $X(s) = cs + d$ for some constant d. Solving $T'(s) = 1$ gives $T(s) = s + f$ for some constant f.

In conclusion $s \mapsto (X(s), T(s)) = (cs + d, s + f)$ is a parametrisation of a line and thus the characteristic curves are indeed straight lines.

(c) Fix $x \in \mathbf{R}$ and consider the characteristic curve $s \mapsto (X(s), T(s))$ which passes though $(x, 0)$ when $s = 0$ (so $X(0) = x$ and $T(0) = 0$). By comparison with what we found in part (a), this means the constant d is equal to x and the constant f is equal to 0. Thus the characteristic curve is $s \mapsto (cs + x, s)$ and it only remains to compute the constant c.

If $x < 0$ then $c = z(0) = u(X(0), T(0)) = u(x, 0) = 0$, so $c = 0$ and the characteristic curve has the form $s \mapsto (x, s)$. This is a vertical straight line.

If $0 \leq x \leq 1$ then $c = z(0) = u(X(0), T(0)) = u(x, 0) = x$, so $c = x$ and the characteristic curve has the form $s \mapsto (xs + x, s)$. This is a straight line in the direction $(x, 1)$.

If $x > 1$ then $c = z(0) = u(X(0), T(0)) = u(x, 0) = 1$, so $c = 1$ and the characteristic curve has the form $s \mapsto (s + x, s)$. This is a straight line in the direction $(1, 1)$.

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2. It's breakfast time for Baby Elwin and his father is warming up his porridge in a saucepan on the hob. Denote the interior of the volume occupied by the porridge by Ω and its boundary by $\partial\Omega$. Neglecting the effects of convection and stirring, the temperature of the porridge u is modelled by the heat equation

$$\partial_t u(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) = 0$$

for $\mathbf{x} \in \Omega$ and $0 < t \leq T$. The initial temperature of the porridge $u(\mathbf{x}, 0)$ ($\mathbf{x} \in \Omega$) is known and Elwin's dad can control the boundary values $u(\mathbf{x}, t)$ for $\mathbf{x} \in \partial\Omega$ and $t > 0$.

(a) Assuming that Ω is a bounded domain and the temperature u is a continuous function in $\overline{\Omega} \times [0, T]$, prove that the highest temperature of the porridge occurs at a point $(\mathbf{x}, t) \in \overline{\Omega} \times [0, T]$ for which either $t = 0$ or $\mathbf{x} \in \partial\Omega$. (That is, prove the weak maximum principle for the heat equation.) **[8 marks]**

If Baby Elwin's dad instead uses the microwave to heat up the porridge the model is modified to the inhomogeneous heat equation

$$\partial_t u(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) = F(\mathbf{x}, t) \quad (\mathbf{x} \in \Omega \text{ and } 0 < t \leq T)$$

where $F(\mathbf{x}, t)$ is controlled by Elwin's dad, but the boundary values of u are now fixed at room temperature.

(b) Do solutions u to the inhomogeneous equation satisfy the same maximum principle from (a) for all possible choices of F? Motivate your answer. **[5 marks]**

(c) On the basis of the phenomena discussed above, which method of heating porridge would you recommend for fathers looking after their children. Briefly motivate your answer. **[3 marks]**

Solution:

(a) For $\varepsilon > 0$ set $v(\mathbf{x}, t) = u(\mathbf{x}, t) + \varepsilon|\mathbf{x}|^2$. As the sum of two continuous functions, v is continuous on $\overline{\Omega} \times [0, T]$ and so must attain a maximum somewhere in the compact set $\overline{\Omega} \times [0, T]$. We will now rule out the possibility that v attains its maximum in $\Omega \times (0, T]$. Suppose to the contrary that v attains this maximum at some point $(\mathbf{x}, t) \in \Omega \times (0, T]$. Then we must have $\partial_t v(\mathbf{x}, t) \geq 0$ and, by the second derivative test, $\Delta v(\mathbf{x}) = \sum_{j=1}^{n} \partial_j^2 v(\mathbf{x}) \leq 0$. Therefore

$$\partial_t v(\mathbf{x}, t) - \Delta v(\mathbf{x}, t) \geq 0 - 0 = 0.$$

But on the other hand, we can compute

$$\partial_t v(\mathbf{x}, t) - \Delta v(\mathbf{x}, t) = \partial_t u(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) - 2\varepsilon n = 0 - 2\varepsilon n < 0,$$

via the differential equation u satisfies. These two inequalities contradict each other, so v cannot attain its maximum in $\Omega \times (0, T]$.

Therefore v must attain its maximum at a point $(\mathbf{y}, s)$ such that either $\mathbf{y} \in \partial\Omega$ or $s = 0$. Thus, for any $(\mathbf{x}, t) \in \overline{\Omega} \times [0, T]$,

$$u(\mathbf{x}, t) \leq v(\mathbf{x}, t) \leq v(\mathbf{y}, s) = u(\mathbf{y}, s) + \varepsilon|\mathbf{y}|^2 \leq u(\mathbf{y}, s) + \varepsilon C^2 \leq \max_{(\partial\Omega \times (0,T]) \cup (\Omega \times \{0\})} u + \varepsilon C^2,$$

where C is the constant obtained from the fact Ω is bounded. Since the above inequality holds for any $\varepsilon > 0$, we have $u(\mathbf{x}, t) \leq \max_{(\partial\Omega \times (0,T]) \cup (\Omega \times \{0\})} u$ for any $(\mathbf{x}, t) \in \overline{\Omega} \times [0, T]$, so

$$\max_{\overline{\Omega} \times [0,T]} u \leq \max_{(\partial\Omega \times (0,T]) \cup (\Omega \times \{0\})} u$$

Because $(\partial\Omega \times (0, T]) \cup (\Omega \times \{0\}) \subseteq \overline{\Omega} \times [0, T]$ the reverse inequality is immediate.

(b) No. To see this we can choose our favourite function which attains a maximum in $\Omega \times (0, T]$ and take this as our function u. With this u in hand we can compute $\partial_t u - \Delta u$ to find the corresponding inhomogeneity F which makes u a solution to the PDE.

(c) The hob is better. As a consequence of the Weak Maximum Principle, the porridge can never be hotter than its initial temperature and the boundary temperature and it is thus relatively easy to ensure Baby Elwin does not burn himself. On the other hand, there is no such maximum principle for the inhomogeneous equation, so the porridge heated in the microwave may have 'hot spots' in the middle, which, being in the middle of the porridge, would be hard for his father to detect before feeding the porridge to Baby Elwin.

3.

Baby Elwin has a toy guitar which both he and his brother Arthur enjoy playing with. One day, while their father is looking after them, Arthur suggests the displacement of the string at a point x along the string's length ℓ at time t is modelled by a function $u(x, t)$ which satisfies the *damped string equation*

$$\partial_{tt} u(x, t) - c^2 \partial_{xx} u(x, t) + r\partial_t u(x, t) = 0 \quad (x \in (0, \ell),\ t > 0)$$

for given constants $c, r > 0$ and the boundary conditions

$$u(0,t) = u(\ell,t) = 0$$

for all $t > 0$.

(a) Show that if we look for a solution of the form $u(x,t) = X(x)T(t)$ (that is, we separate variables) the functions X and T must satisfy

$$c^2 X''(x) + \lambda X(x) = 0$$

and

$$T''(t) + rT(t) + \lambda T(t) = 0$$

respectively, for some constant λ. **[4 marks]**

(b) Show that the boundary conditions imply λ must be taken to be positive if we wish to find a non-zero solution via this method. Calculate the precise values λ can take on. **[4 marks]**

(c) Show that the function T obtained via this method decays exponentially in t provided the damping coefficient r satisfies $r^2/4 > \lambda$. **[4 marks]**

(d) Without doing any further precise calculations, describe how the solution will decay if $r^2/4 < \lambda$. **[4 marks]**

Solution:

(a) Substituting the ansatz $u(x,t) = X(x)T(t)$ into the equation, we obtain

$$X(x)T''(t) - c^2 X''(x)T(t) + rX(x)T'(t) = 0$$

Dividing by $X(x)T(t)$ we obtain

$$\frac{T''(t)}{T(t)} - \frac{c^2 X''(x)}{X(x)} + \frac{rT'(t)}{T(t)} = 0$$

rewriting this as

$$\frac{T''(t)}{T(t)} + \frac{rT'(t)}{T(t)} = \frac{c^2 X''(x)}{X(x)}$$

we can see that the left-hand side cannot depend on x and the right-hand side cannot depend on t. Since the two sides are equal, we deduce that they are both constant. Let's denote this constant by $-\lambda$. Thus

$$\frac{T''(t)}{T(t)} + \frac{rT'(t)}{T(t)} = -\lambda = \frac{c^2 X''(x)}{X(x)}.$$

The two equations easily follow from here.

(b) The boundary conditions say that $X(0)T(t) = X(\ell)T(t) = 0$ for all $t > 0$. Since $T \not\equiv 0$ we must have $X(0) = X(\ell) = 0$.

We now solve the ODE involving X. The ODEs characteristic polynomial is $p(z) = c^2 z^2 + \lambda$. From this we can see that if $\lambda \leq 0$ the solution X will be either an exponential or a linear function. Such functions cannot satisfy $X(0) = X(\ell) = 0$ without being identically zero. Thus we require $\lambda > 0$.

For $\lambda > 0$ the characteristic polynomial has the roots $i\sqrt{\lambda}/c$ and $-i\sqrt{\lambda}/c$, which leads to the general solution

$$X(s) = A\cos(\sqrt{\lambda}s/c) + B\sin(\sqrt{\lambda}s/c)$$

for constants A and B. To satisfy $X(0) = 0$ we require $A = 0$ and to satisfy $X(\ell) = 0$ we can only take λ equal to $\lambda_k = \pi^2 k^2 c^2$ where k is a positive integer.

(c) We now wish to solve

$$T''(t) + rT(t) + \lambda_k T(t) = 0$$

with $\lambda_k = \pi^2 k^2 c^2$ and $r^2/4 > \lambda_k$. The characteristic polynomial for this ODE is

$$q(z) = z^2 + rz + \lambda_k = \left(z + \frac{r}{2}\right)^2 + \lambda_k - \frac{r^2}{4}$$

Thus, since $r^2/4 - \lambda_k > 0$, the roots of q are

$$z_\pm = -\frac{r}{2} \pm \sqrt{\frac{r^2}{4} - \lambda_k}.$$

Clearly z_- is negative, but, since λ_k is positive, $\sqrt{\frac{r^2}{4} - \lambda_k} < \frac{r}{2}$, so even z_+ is negative. This means that T will decay exponentially.

(d) If $r^2/4 < \lambda$ the polynomial q's discriminant becomes negative and we obtain complex-valued roots. However the real part is always $-r/2 < 0$, so T both decays exponentially and oscillates.

4.

Baby Elwin and his brother, Arthur, are playing on an old trampoline. The frame is circular in shape, but, due to the unevenness of the ground beneath it, it has bent so that its height varies around the edge of the trampoline. Arthur produces a simple model for the stationary taut trampoline before they climb on it: The height of the trampoline above a point $\mathbf{x}$ on the ground is given by the solution $u(\mathbf{x})$ of the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } D_a, \text{ and} \\ u = h & \text{on } \partial D_a, \end{cases}$$

where $D_a = \{\mathbf{x} \in \mathbf{R}^2 \mid |\mathbf{x}| < a\}$ is the disc of radius a above which the trampoline lies.

Arthur even manages to derive the Poisson formula for the solution u:

$$u(\mathbf{x}) = \frac{(a^2 - |\mathbf{x}|^2)}{2\pi a} \int_{|\mathbf{y}|=a} \frac{h(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2} d\sigma(\mathbf{y}).$$

(a) For a harmonic function $u\colon \Omega \to \mathbf{R}$, where $\Omega \subset \mathbf{R}^2$ is an open set, state the mean value property u satisfies. **[4 marks]**

(b) Use the Poisson formula above to prove the mean value property you stated in part (a). **[6 marks]**

(c) Suppose $u\colon \overline{D_a} \to \mathbf{R}$ is harmonic in D_a and such that $u(a, \theta) = \sin(4\theta) + 2$ on ∂D_a (written in polar coordinates). Without calculating the solution explicitly, compute the maximum value of u in $\overline{B_a}$ and the value of u at the origin. **[6 marks]**

Solution:

(a) **Theorem.** Let $\Omega \subset \mathbf{R}^2$ be an open set and $u\colon \Omega \to \mathbf{R}$ harmonic. Then for any disc $D_a(\mathbf{x}) = \{\mathbf{y} \in \mathbf{R}^2 \mid |\mathbf{y} - \mathbf{x}| < a\}$ such that $\overline{D_a(\mathbf{x})} \subset \Omega$ we have that

$$u(\mathbf{x}) = \frac{1}{2\pi a} \int_{\partial D_a(\mathbf{x})} u(\mathbf{y}) d\sigma(\mathbf{y}).$$

That is, the value of u at the centre of a disc is equal to the mean value of u over the boundary of the disc.

(b) We apply the Poisson formula to the function $u(\cdot + \mathbf{x})$, which is harmonic in D_a and has boundary values $h(\mathbf{y}) = u(\mathbf{y} + \mathbf{x})$. Evaluating this formula at the centre of the disc gives

$$u(\mathbf{0}+\mathbf{x}) = \frac{a^2}{2\pi a} \int_{|\mathbf{y}|=a} \frac{h(\mathbf{y})}{|\mathbf{0} - \mathbf{y}|^2} d\sigma(\mathbf{y}) = \frac{a^2}{2\pi a} \int_{|\mathbf{y}|=a} \frac{u(\mathbf{y} + \mathbf{x})}{a^2} d\sigma(\mathbf{y}) = \frac{1}{2\pi a} \int_{|\mathbf{y}|=a} u(\mathbf{y}+\mathbf{x}) d\sigma(\mathbf{y}).$$

This is exactly the mean value property.

(c) The maximum must be attained at the boundary, and the maximum boundary value is 3, so this is also the maximum value of u in $\overline{B_a}$. The value $u(\mathbf{0})$ is the mean of the values $u(a, \theta)$ over $\theta \in (0, 2\pi)$, which is 2.

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5. Baby Elwin has trouble understanding derivatives, so prefers to use finite differences instead, saying that he heard somewhere they are more or less the same. The aim of this question is to convince his brother, Arthur, that, in some sense, Baby Elwin is justified in conflating the two notions.

Fix $\delta x > 0$. For a function $u\colon \mathbf{R} \to \mathbf{R}$ which is three times continuously differentiable, define $x_j = j\delta x$ and set $u_j = u(x_j)$.

(a) Define the forward, backward and centred difference of u at x_j. **[6 marks]**

(b) Use Taylor's theorem to estimate the error between the differences you defined above and the first derivative $u'(x_j)$. State clearly any assumptions you make on the function u. **[10 marks]**

Solution:

(a)

$$\text{The backward difference:} \quad \frac{u_j - u_{j-1}}{\delta x}.$$

$$\text{The forward difference:} \quad \frac{u_{j+1} - u_j}{\delta x}.$$

$$\text{The centred difference:} \quad \frac{u_{j+1} - u_{j-1}}{2\delta x}.$$

(b) If $u \in C^3(\mathbf{R})$ then

$$u(x+h) = u(x) + u'(x)h + \frac{u''(x)}{2}h^2 + O(h^3). \tag{1}$$

Taking $x = x_j$ and $h = -\delta x$ we see that

$$u'(x_j) = \frac{u_j - u_{j-1}}{\delta x} + O(\delta x),$$

taking $x = x_j$ and $h = \delta x$ we see

$$u'(x_j) = \frac{u_{j+1} - u_j}{\delta x} + O(\delta x),$$

and finally taking the difference of (1) with first $x = x_j$ and $h = -\delta x$, and then $x = x_j$ and $h = \delta x$, we obtain

$$u'(x_j) = \frac{u_{j+1} - u_{j-1}}{2\delta x} + O((\delta x)^2).$$
