Instructions: Please attempt all questions. You may answer either in English or Swedish. There are five questions, each worth 16 points. To obtain a grade 3, 4 or 5, you must obtain at least 40, 48 or 56 points (50%, 60% or 70%) respectively. You may not use any notes, textbooks or electronic devices. Good luck!

Svara på alla uppgifter. Du får svara antingen på engelska eller svenska. Det finns fem uppgifter och varje uppgift kan ge maximalt 16 poäng. För att få betyg 3, 4 eller 5 krävs minst 40, 48 respektive 56 poäng (50%, 60% respektive 70%). Inga hälpmedel tillåtna. Lycka till!

(1) Let $f\colon \mathbf{R}^n \times (0,\infty) \to \mathbf{R}$ and $g\colon \mathbf{R}^n \to \mathbf{R}$ be two smooth functions and $b \in \mathbf{R}^n$. Consider the equations

$$\begin{aligned} u_t(x,t) + b\cdot\nabla u(x,t) = f(x,t) \quad &\text{for } x \in \mathbf{R}^n \text{ and } t > 0, \text{ and} \\ u(x,0) = g(x) \quad &\text{for } x \in \mathbf{R}^n. \end{aligned} \qquad (\clubsuit)$$

Here ∇ denotes the gradient vector in the x-variables.

(a) Show that the characteristic curves (X,T) of the homogeneous equation $v_t(x,t) + b\cdot\nabla v(x,t) = 0$ are given by $X(s) = x + bs$ and $T(s) = t + s$. **[4 marks]**

(b) Let u be a solution to $(\clubsuit)$. Set $z(s) = u(x+bs, t+s)$ for fixed $x \in \mathbf{R}^n$ and $t > 0$ and derive an ordinary differential equation which z satisfies. **[6 marks]**

(c) Use the ordinary differential equation from (b) to derive the formula

$$u(x,t) = g(x-bt) + \int_0^t f(x-bs, t-s)ds.$$

for a solution u to $(\clubsuit)$. **[6 marks]**

(2) (a) State Green's first identity. **[6 marks]**

(b) Let Ω be an open set with C^1 boundary and $h\colon \partial\Omega \to \mathbf{R}$ a C^1 function. Define the energy of each continuously differentiable $v\colon \Omega \to \mathbf{R}$ to be

$$E_h[v] = \frac{1}{2}\int_\Omega |\nabla v(\mathbf{x})|^2 d\mathbf{x} - \int_{\partial\Omega} h(\mathbf{x})v(\mathbf{x})d\sigma(\mathbf{x}).$$

Show that a function $u \in C^2(\overline{\Omega})$ which satisfies the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \text{ and} \\ \frac{\partial u}{\partial \mathbf{n}} := \mathbf{n}\cdot\nabla u = h & \text{on } \partial\Omega \end{cases}$$

is such that

$$E_h[u] \le E_h[v]$$

for all $v \in C^1(\overline{\Omega})$. Here $\mathbf{n}$ is the outward unit normal to $\partial\Omega$. **[10 marks]**

(3) Laplace's equations $\Delta u = 0$ in polar coordinates (r, θ) takes the form

$$\frac{\partial^2 u}{\partial r^2}(r,\theta) + \frac{1}{r}\frac{\partial u}{\partial r}(r,\theta) + \frac{1}{r^2}\frac{\partial u}{\partial \theta}(r,\theta) = 0.$$

(a) By assuming a solution to Laplace's equation u is of the form $u(r,\theta) = R(r)\Theta(\theta)$, derive the two separate equations

$$\Theta'' + \lambda\Theta = 0 \quad \text{and} \quad r^2R'' + rR' - \lambda R = 0$$

for R and Θ, where $\lambda \in \mathbf{R}$. **[2 marks]**

(b) Solve the equation for Θ for each $\lambda \in \mathbf{R}$. If we require that Θ is 2π-periodic what restriction does this impose on λ? **[5 marks]**

(c) Just for those λ which produce a periodic Θ solve the equation for R. If we require u to be continuous at the origin, which possible R will we reject? **[5 marks]**

(d) Assume now we are interested in solutions u defined on $\{(r,\theta) \,|\, 0 \leq r \leq 1\}$. By forming linear combinations of the products $u(r,\theta) = R(r)\Theta(\theta)$ derived above, write down a series which solves Laplace's equation for $r < 1$ and is continuous at the origin if the chosen coefficients are uniformly bounded. Using your knowledge of Fourier series, make an educated guess for a good choice of these coefficients if we wish to impose the boundary condition $u(1,\theta) = h(\theta)$ for a given continuous 2π-periodic function $h\colon \mathbf{R} \to \mathbf{R}$. **[4 marks]**

(4) By factorising the differential operator which appears on the left-hand side of the PDE solve the following initial value problems.

(a)

$$\begin{cases} \partial_{tt}u(x,t) - 3\partial_{xt}u(x,t) - 4\partial_{xx}u(x,t) = 0 & \text{for } x \in \mathbf{R} \text{ and } t > 0, \\ u(x,0) = \arctan x \quad \text{and} \quad \partial_t u(x,0) = e^{-x^2} & \text{for } x \in \mathbf{R}. \end{cases}$$

[8 marks]

(b)

$$\begin{cases} \partial_{tt}u(x,t) + \partial_{xt}u(x,t) - 20\partial_{xx}u(x,t) = 0 & \text{for } x \in \mathbf{R} \text{ and } t > 0, \\ u(x,0) = x^2 \quad \text{and} \quad \partial_t u(x,0) = \sin x & \text{for } x \in \mathbf{R}. \end{cases}$$

[8 marks]

(5) Consider a function $u\colon \mathbf{R} \to \mathbf{R}$ of one variable. For a given mesh size δx define $x_j = j(\delta x)$ and

$$u_j = u(x_j).$$

(a) Using the above notation (if you wish) define the three standard approximations to the first derivative of u: the backward difference, forward difference and centred difference. **[4 marks]**

(b) Prove that the error between the derivative of u at x_j and the backward difference is $O(\delta x)$. **[4 marks]**

(c) Prove that the error between the derivative of u at x_j and the forward difference is $O(\delta x)$. **[4 marks]**

(d) Prove that the error between the derivative of u at x_j and the centred difference is $O((\delta x)^2)$. **[4 marks]**