Instructions: Please attempt all questions. You may answer either in English or Swedish. There are five questions, each worth 16 points. To obtain a grade 3, 4 or 5, you must obtain at least 40, 48 or 56 points (50%, 60% or 70%) respectively. You may not use any notes, textbooks or electronic devices. Good luck!

Svara på alla uppgifter. Du får svara antingen på engelska eller svenska. Det finns fem uppgifter och varje uppgift kan ge maximalt 16 poäng. För att få betyg 3, 4 eller 5 krävs minst 40, 48 respektive 56 poäng (50%, 60% respektive 70%). Inga hälpmedel tillåtna. Lycka till!

(1) Let $g\colon \mathbf{R} \to \mathbf{R}$ be a smooth function. Consider the equations

$$\begin{aligned} y^2 u_x(x,y) + \frac{1+y^2}{y} u_y(x,y) &= 0 \quad \text{for } x \in \mathbf{R} \text{ and } y > 0, \text{ and} \\ u(x,0) &= g(x) \quad \text{for } x \in \mathbf{R}^n. \end{aligned} \tag{$\clubsuit$}$$

(a) Show that characteristic curves $t \mapsto (X(t), Y(t))$ of ($\clubsuit$), which are defined to be curves along which a solution u of ($\clubsuit$) is constant, can be chosen to satisfy

$$X'(t) = Y(t)^2 \quad \text{and} \tag{$\diamondsuit$a}$$

$$Y'(t) = \frac{1+Y(t)^2}{2Y(t)}. \tag{$\diamondsuit$b}$$

[3 marks]

(b) Show that all the solutions to ($\diamondsuit$b) are of the form $Y(t) = \sqrt{Ae^t - 1}$ for a constant $A \in \mathbf{R}$. Find all the solutions X of ($\diamondsuit$a). **[6 marks]**

(c) For fixed $x \in \mathbf{R}$ and $y > 0$, choose X and Y from those you found in (1b) so that $X(0) = x$ and $Y(0) = y$. With this choice of Y find $s \in \mathbf{R}$ such that $Y(s) = 0$? What is $X(s)$? **[4 marks]**

(d) Using your answers from above, find a solution to ($\clubsuit$). **[3 marks]**

(2) (a) State Green's first identity. **[5 marks]**

(b) Let Ω be an open set with C^1 boundary, and let $f\colon \Omega \to \mathbf{R}$. Use Green's first identity to prove the uniqueness of solutions $u \in C^2(\Omega)$ to the following boundary value problem:

$$\begin{cases} \Delta u = f & \text{in } \Omega, \text{ and} \\ u = h & \text{on } \partial\Omega. \end{cases}$$

[6 marks]

(c) Green's second identity says that for two functions $u, v \in C^2(\overline{\Omega})$

$$\int_\Omega u(\mathbf{x})\Delta v(\mathbf{x}) - v(\mathbf{x})\Delta u(\mathbf{x}) d\mathbf{x} = \int_{\partial\Omega} u(\mathbf{x})\frac{\partial v}{\partial \mathbf{n}}(\mathbf{x}) - v(\mathbf{x})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) d\sigma(\mathbf{x}).$$

Use Green's first identity to prove Green's second identity. **[5 marks]**

(3) The aim of this question is to solve the initial value problem

$$\begin{cases} \partial_t u(x,t) - \partial_{xx} u(x,t) = 0 & \text{for } x \in \mathbf{R} \text{ and } t \in (0,\infty); \\ u(x,0) = \phi(x) & \text{for } x \in \mathbf{R}, \end{cases} \qquad (\heartsuit)$$

with initial data

$$\phi(x) = \begin{cases} 1 & \text{if } x > 0; \\ \frac{1}{2} & \text{if } x = 0; \\ 0 & \text{if } x < 0. \end{cases}$$

(a) We begin by looking for a solution of the form

$$u(x,t) = g(x/(2\sqrt{t}))$$

for some $g\colon \mathbf{R} \to \mathbf{R}$. Show that if any such u solves ($\heartsuit$) then g solves the ordinary differential equation

$$g''(p) + 2pg'(p) = 0.$$

[6 marks]

(b) Find the general formula for solutions g of the ordinary differential equation above. Use this formula for g together with the initial data ϕ to find the solution u of ($\heartsuit$). You may use the fact $\int_0^\infty e^{-q^2}\,dq = \int_{-\infty}^0 e^{-q^2}\,dq = \sqrt{\pi}/2$ without proof. **[7 marks]**

(c) Observe that $\partial_x u(x,t)$ is equal to $S(x,t)$, the heat kernel. Can you give some motivation for why this is so? **[3 marks]**

(4) Consider the function $\Phi\colon \mathbf{R}^n \setminus \{\mathbf{0}\} \to \mathbf{R}$ defined by

$$\Phi(\mathbf{x}) = \begin{cases} -\frac{1}{2\alpha(2)} \ln|\mathbf{x}| & \text{if } n = 2, \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|\mathbf{x}|^{n-2}} & \text{if } n > 2, \end{cases}$$

where $\alpha(n)$ is the volume of the unit ball in $\mathbf{R}^n$. (So, in particular, $\alpha(2) = \pi$ and $\alpha(3) = 4\pi/3$.)

(a) Prove that Φ is harmonic on $\mathbf{R}^n \setminus \{\mathbf{0}\}$. **[4 marks]**

(b) Consider the domain $B_r(\mathbf{0}) = \{\mathbf{y} \in \mathbf{R}^n \,|\, |\mathbf{y}| < r\}$. Then the outward unit normal at $\mathbf{x} \in \partial B_r(\mathbf{0})$ is $\mathbf{n}(\mathbf{x}) = \mathbf{x}/|\mathbf{x}|$. Prove that

$$\frac{\partial \Phi}{\partial \mathbf{n}}(\mathbf{x}) = \frac{-1}{n\alpha(n)} \frac{1}{|\mathbf{x}|^{n-1}}$$

for each $n = 1, 2, \ldots$. **[6 marks]**

(c) Let Ω be an open bounded set with C^1 boundary and suppose that $u \in C^2(\overline{\Omega})$ is harmonic in Ω. Apply Green's second identity to $\Omega_r = \Omega \setminus B_r(\mathbf{x})$ (where $B_r(\mathbf{x}) = \{\mathbf{y} \in \mathbf{R}^n \,|\, |\mathbf{y} - \mathbf{x}| < r\}$) to prove that

$$u(\mathbf{x}) = \int_{\partial\Omega} \left\{ \Phi(\mathbf{y} - \mathbf{x}) \left(\frac{\partial u}{\partial \mathbf{n}} \right)(\mathbf{y}) - \left(\frac{\partial \Phi}{\partial \mathbf{n}} \right)(\mathbf{y} - \mathbf{x}) u(\mathbf{y}) \right\} d\sigma(\mathbf{y}).$$

You may assume $\int_{\partial B_r(\mathbf{x})} d\sigma = n\alpha(n) r^{n-1}$ without proof. **[6 marks]**

(5) Consider the initial boundary value problem

$$\begin{cases} \partial_{tt}v(x,t) - \partial_{xx}v(x,t) = 0 & \text{for } x \in (0,\ell) \text{ and } t \in (0,T], \\ v(x,0) = g(x) \quad \text{and} \quad \partial_t v(x,0) = h(x) & \text{for } x \in [0,\ell], \text{ and} \\ v(0,t) = 0 \quad \text{and} \quad v(\ell,t) = 0 & \text{for } t \in (0,T]. \end{cases} \qquad (\spadesuit)$$

(a) If v solved the heat equation instead of the wave equation, then v would satisfy a weak maximum principle. State the weak maximum principle for the heat equation. **[8 marks]**

(b) Find a specific choice of functions g and h, and $T > 0$, together with a solution v to ($\spadesuit$), which prove that such a weak maximum principle for the wave equation is false. Explain clearly why your solution proves there is no maximum principle for the wave equation. **[8 marks]**