1 Homework 7: Fixed Point Theorems

We are interested in the effects of demographics on siberian musk deers and want to study the effects of vital rates that are periodic in time. Taking this into consideration we will use the population model developed by von Foerster and generalise it by making the vital rates time dependent. We have the following balance equation:

$$\frac{\partial n(a,t)}{\partial t} + \frac{\partial n(a,t)}{\partial a} = -\mu(a,t)n(a,t), \quad a,t > 0 \tag{1}$$

and the boundary and initial conditions are given by

$$n(0,t) = \int_0^\infty m(a,t)n(a,t)da, \quad t > 0 \tag{2}$$

$$n(a,0) = f(a), \quad a \geq 0 \tag{3}$$

where $n_t(a) = n(a,t)$ represent the age distribution of the population at time t, the death rate and birth rate are age and time dependent and denoted $\mu(a,t)$ and $m(a,t)$ respectively and $f(a)$ is the initial age distribution of the population.

We assume that there is an upper bound on the age of individuals and denote this bound by A_μ. The individuals in a population also have a constant lower and upper bound on the fertile phase of individuals. We assume that the functions $m(a,t), \mu(a,t)$ and $f(a)$ satisfies the following properties:

(i) $m(a,t)$ is bounded for $a,t \geq 0$
$m(a,t) = 0$ for $a > A_\mu$ and $t \geq 0$
$m(a,t) \geq \delta_1 > 0$ for $a_1 < a < a_2$, where $0 < a_1 < a_2 < A_\mu$ and $t \geq 0$

(ii) $0 < c_\mu \leq \mu(a,t) \leq C_\mu < \infty$ for $a \leq a_2$ and $t \geq 0$
$\int_A^{A+A_\mu} \mu(a,t)da = \infty$ for $t \geq 0$ and $A \geq 0$

(iii) f is bounded
$f(a) \geq \delta_2 > 0$ for $0 < a < a_2$
$f(a) = 0$ for $a > A_\mu$

Using the method of characteristics one can deduce the following

Theorem 1.1. *Let $n(a,t)$ be a solution to (1) with boundary and initial condition (2) and (3). Then*

$$\begin{aligned} n(0,t) = &\int_0^t m(a,t)e^{-\int_0^a \mu(v,v+t-a)dv} n(0,t-a)da \\ &+ \int_t^\infty m(a,t)e^{-\int_{a-t}^a \mu(v,v+t-a)dv} f(a-t)da \end{aligned} \tag{4}$$

for $t > 0$. For $a > 0$, $n(a,t)$ is given by

$$n(a,t) = \begin{cases} n(0,t-a)e^{-\int_0^a \mu(v,v+t-a)dv}, & a < t \\ f(a-t)e^{-\int_{a-t}^a \mu(v,v+t-a)dv}, & a \geq t \end{cases}. \tag{5}$$

We denote $L^\infty_\Lambda(0,\infty)$ the space of measurable functions u on $[0,\infty)$ that satisfy $|u(t)|= O(e^{\Lambda t})$ for $t \geq 0$ where Λ is a positive real number. The norm $||u||_\Lambda$ on $L^\infty_\Lambda(0,\infty)$ is defined by

$$||u||_\Lambda = \sup_{t>0} |u(t)| e^{-\Lambda t}.$$

It is a Banach space for every real Λ.

Definition 1.1. A function $n_+ \in X$ is an upper solution to the fixed point equation $A[u] = u$ if

$$n_+(t) \geq A[n_+](t)$$

for all $t > 0$. Similarly a function $n_- \in X$ is a lower solution to the fixed point equation $A[u] = u$ if

$$n_-(t) \leq A[n_-](t)$$

for all $t > 0$.

Exercise 1.a

Let X be a Banach space of functions over $\mathbb{R}^n$. Let $A : X \to X$ be a monotonically increasing (meaning if $u, v \in X$ and $u \leq v$ then $A(u) \leq A(v)$. $u \leq v$ means that $u(x) \leq v(x)$ for all x) mapping that is also a strict contraction. Let $n_0 = n^+$ be an upper solution and thereafter iteratively define $n_{k+1} = A[n_k]$. Show with the help of Evans 9.2.1 that $n_0 \geq n_1 \geq \dots n^*$ where n^* is the unique fixed point to A. In particular we have $n^+ \geq n^*$. In the same way one can conclude that $n^- \leq n^*$ where n^- is a lower solution.

Exercise 1.b

For brevity we use the notation

$$Q(a,t) = m(a,t)e^{-\int_0^a \mu(v,v+t-a)dv}, \quad a \geq 0,\, t \geq 0. \tag{6}$$

Show that (4) has a unique solution in $L^\infty_\Lambda(0,\infty)$ for large enough Λ. Show also that if n_+ and n_- are upper and lower solutions to (4) then the "number of newborn siberian musk deers" $n(0,t)$ satisfies $n_-(t) \leq n(0,t) \leq n_+(t)$.

This homework is based upon the article "Estimating effective boundaries of population growth in a variable environment"
Estimating effective boundaries that can be found in Sonja Radosavljevic's thesis:
"Permanence of age-structured populations in spatio-temporal variable environment