Graduate Course in Partial Differential Equations (MAI0133)
Spring Semester 2017
Homework 2

The questions below are taken from Evans. Here we will assume that U is a bounded connected set with a smooth boundary — this is more than we need, but we are not worried about making minimal assumptions. A step in the proof of the energy estimates I skipped over in the lecture makes use of Poincaré's inequality. This can be stated in slightly different forms. The two which are of interest to us here are

$$\|u\|_{L^2(U)} \leq C\|Du\|_{L^2(U)}$$

for all $u \in H_0^1(U)$ and

$$\|u - (u)_U\|_{L^2(U)} \leq C\|Du\|_{L^2(U)}$$

for all $u \in H^1(U)$, where $(u)_U = \frac{1}{|U|}\int_U u$ is the average of u over U. You may use them without proof, but look in Evans if you want to know more.

2.1 Assume $a^{ij}, c \in L^\infty(U)$ for all $i, j = 1, \ldots, n$ and $(a^{ij})_{ij}$ satisfies the ellipticity condition $\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2$ for some $\theta > 0$ and all $\xi = (\xi_1, \ldots, \xi_n)$ and x in $\mathbf{R}^n$. Let

$$Lu = -\sum_{i,j=1}^n (a^{ij}u_{x_i})_{x_j} + cu.$$

Prove that there exists a constant $\mu > 0$ such that the corresponding bilinear form $B[\cdot,\cdot]$ satisfies the hypothesis of the Lax-Milgram Theorem provided $c(x) \geq -\mu$ $(x \in U)$.

2.2 A function $u \in H^1(U)$ is said to be a *weak solution* of the *Neumann problem*

$$\begin{cases} -\Delta u = f & \text{in } U \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial U \end{cases} \qquad (*)$$

if

$$\int_U Du(x) \cdot Dv(x)dx = \int_U f(x)v(x)dx$$

for all $v \in H^1(U)$. Suppose $f \in L^2(U)$. Prove that $(*)$ has a weak solution if and only if

$$\int_U f(x)dx = 0.$$

Please feel free to look through Chapter 5 and 6 in Evans for supporting theory and further reading. I am happy to also discuss your thoughts/problems related to the theory or any other questions you decide to think about related to the course.