Geometry
Second Semester 2010/11
Homework 1

1. Let $\mathbf{v} = (p, v) = (p, v_1, v_2)$ and $\mathbf{w} = (p, w) = (p, w_1, w_2)$ be two vectors based at the point $p \in \mathbf{R}^2$ and denote by θ the angle between them. Use your knowledge of trigonometry (or any other method) to prove that
$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos\theta,$$
where $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2$ and $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$.

2. Find the local minima and local maxima of the following functions. Justify the answers you obtain in each case.
   (a) $f\colon \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^3 - 2x^2 + x - 3$ for all $x \in \mathbf{R}$.
   (b) $g\colon (-3, 3] \to \mathbf{R}$ defined by $g(x) = x^3 - 2x^2 + x - 3$ for all $x \in (-3, 3]$.
   (c) $h\colon \mathbf{R}^2 \to \mathbf{R}$ defined by $h(x_1, x_2) = x_1^2 - 4x_1 + 4 + x_2^2$ for all $(x_1, x_2) \in \mathbf{R}^2$.
   (d) $k\colon \mathbf{R}^2 \to \mathbf{R}$ defined by $k(x_1, x_2) = x_1^2 - x_2^2$.

3. Prove the chain rule for real-valued functions: Let $f\colon (a, b) \to \mathbf{R}$ be differentiable at $x \in (a, b)$ and such that $f(x) \in (c, d)$. Also let $G\colon (c, d) \to \mathbf{R}$ be differentiable at $f(x)$. Prove that $G \circ f$ is differentiable at x and
$$(G \circ f)'(x) = G'(f(x))f'(x).$$

4. Sketch typical level sets and the graph of the function $f\colon \mathbf{R}^2 \to \mathbf{R}$ when defined as follows.
   (a) $f(x_1, x_2) = x_1$ for all $(x_1, x_2) \in \mathbf{R}^2$.
   (b) $f(x_1, x_2) = x_1 - x_2$ for all $(x_1, x_2) \in \mathbf{R}^2$.
   (c) $f(x_1, x_2) = x_1^2 - x_2^2$ for all $(x_1, x_2) \in \mathbf{R}^2$.

5. Sketch the level sets $f^{-1}(c)$ for $n = 0, 1$ and 2 of the function $f\colon \mathbf{R}^{n+1} \to \mathbf{R}$ when defined as follows at the heights indicated.
   (a) $f(x_1, x_2, \ldots, x_{n+1}) = x_{n+1}$ for all $(x_1, x_2, \ldots, x_{n+1}) \in \mathbf{R}^{n+1}$; $c = -1, 0, 1, 2$.
   (b) $f(x_1, x_2, \ldots, x_{n+1}) = 0x_1^2 + x_2^2 + \ldots + x_{n+1}^2$ for all $(x_1, x_2, \ldots, x_{n+1}) \in \mathbf{R}^{n+1}$; $c = 0, 1, 4$.
   (c) $f(x_1, x_2, \ldots, x_{n+1}) = x_1 - x_2^2 + \ldots - x_{n+1}^2$ for all $(x_1, x_2, \ldots, x_{n+1}) \in \mathbf{R}^{n+1}$; $c = -1, 0, 1, 2$.
   (d) $f(x_1, x_2, \ldots, x_{n+1}) = x_1^2 - x_2^2 + \ldots - x_{n+1}^2$ for all $(x_1, x_2, \ldots, x_{n+1}) \in \mathbf{R}^{n+1}$; $c = -1, 0, 1$.

6. (a) Show that the graph of any function $f\colon \mathbf{R}^n \to \mathbf{R}$ is a level set for some function $F\colon \mathbf{R}^{n+1} \to \mathbf{R}$.
   (b) Give an example of a level set which is not the graph of a function. Explain why the example you give cannot be the graph of a function.