Geometry
Second Semester 2011/12
Homework 4

1. For what values of c is the level set $f^{-1}(c)$ an n-surface for each f given below.

   (a) $f(x_1, x_2, \ldots, x_{n+1}) = x_1^2 + x_2^2 + \ldots + x_{n+1}^2$
   (b) $f(x_1, x_2, \ldots, x_{n+1}) = x_1^2 + x_2^2 + \ldots + x_n^2 - x_{n+1}^2$
   (c) $f(x_1, x_2, \ldots, x_{n+1}) = x_1 x_2 \ldots x_{n+1} + 1$

2. Show that the cylinder $\{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 = 1\} \subset \mathbf{R}^3$ can be represented as the level set of each of the following functions.

   (a) $f(x_1, x_2, x_3) = x_1^2 + x_2^2$
   (b) $f(x_1, x_2, x_3) = -x_1^2 - x_2^2$
   (c) $f(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + \sin(x_1^2 + x_2^2)$

3. Show that if an n-surface S is represented as both $f^{-1}(c)$ and $g^{-1}(d)$, for smooth functions f and g, and numbers $c, d \in \mathbf{R}$, where $\nabla f(p) \neq \mathbf{0}$ and $\nabla g(p) \neq \mathbf{0}$ for all $p \in S$, then, for each $p \in S$, $\nabla f(p) = \lambda \nabla g(p)$ for some real number $\lambda \neq 0$.

4. Sketch the cylinders $f^{-1}(0)$ for the following functions f.

   (a) $f(x_1, x_2) = x_1$
   (b) $f(x_1, x_2, x_3) = x_1 - x_2^2$
   (c) $f(x_1, x_2, x_3) = x_1^2/4 + x_2^2/9 - 1$

5. Verify that a surface of revolution (see Example 4.6 in your notes) is a 2-surface.

6. Sketch the surface of revolution obtained by rotating C about the x_1-axis, where C is each of the curves defined below.

   (a) $\{(x_1, x_2) \mid x_2 = 1\}$ (cylinder)
   (b) $\{(x_1, x_2) \mid -x_1^2 + x_2^2 = 1, x_2 > 0\}$ (1-sheeted hyperboloid)
   (c) $\{(x_1, x_2) \mid x_1^2 + (x_2 - 2)^2 = 1\}$ (torus)

7. Show that the set S of all unit vectors at all points in $\mathbf{R}^2$ forms a 3-surface in $\mathbf{R}^4$. Hint: $S = \{(x_1, x_2, x_3, x_4) \mid x_3^2 + x_4^2 = 1\}$.

8. Let $S = f^{-1}(c)$ be a 2-surface in $\mathbf{R}^3$ which lies in the half-space $\{(x_1, x_2, x_3) \mid x_3 > 0\}$. Find a function $g: U \to \mathbf{R}$ (where U an open set in $\mathbf{R}^4$) such that $g^{-1}(c)$ is the 3-surface obtained by rotating the 2-surface S about the (x_1, x_2)-plane.

9. Let $a, b, c \in \mathbf{R}$ be such that $ac - b^2 > 0$. Show that the maximum and minimum values of the function $g(x_1, x_2) = x_1^2 + x_2^2$ on the ellipse $\{(x_1, x_2) \mid ax_1^2 + 2bx_1x_2 + cx_2^2 = 1\}$ are of the form $1/\lambda_1$ and $1/\lambda_2$, where λ_1 and λ_2 are the eigenvalues of the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$.

10. Let S be an n-surface in $\mathbf{R}^{n+1}$ and let $p_0 \in \mathbf{R}^{n+1} \setminus S$. Show that the shortest line segment from p_0 to S (if one exists) is perpendicular to S. That is, show that if $p \in S$ is such that $\|p_0 - p\|^2 \leq \|p_0 - q\|^2$ for all $q \in S$, then $(p, p_0 - p) \perp S_p$. [Hint: Use the Lagrange Multiplier Theorem.] Show that the same conclusion holds for the longest line segment from p_0 to S (again, if one exists).

11. The set $\mathbf{R}^4$ may be viewed as the set of all 2×2 matrices with real entries by identifying the quadruple (x_1, x_2, x_3, x_4) with the matrix

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

The subset consisting of those matrices with determinant equal to one forms a group under matrix multiplication, this group is called the special linear group $SL(2)$. Show that $SL(2)$ is a 3-surface in $\mathbf{R}^4$.

12. The trace of a matrix

$$A = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

is defined to be $\mathrm{tr}(A) := x_1 + x_4$. Show that the tangent space $SL(2)_p$ to $SL(2)$ (see Question 11) at $p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ can be identified with the set of all 2×2 matrices of trace zero by showing that

$$SL(2)_p = \left\{ \left(p, \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \right) \Big| \, x_1 + x_4 = 0 \right\}.$$

[Hint: First show that if

$$\alpha(t) = \begin{pmatrix} x_1(t) & x_2(t) \\ x_3(t) & x_4(t) \end{pmatrix}$$

is a parametrised curve in $SL(2)$ with $\alpha(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $x_1'(0) + x_4'(0) = 0$. Use a dimensional argument to prove the opposite inclusion.]