**Geometry**
**Second Semester 2011/12**
The Existence and Uniqueness of Solutions to
First-Order Ordinary Differential Equations.

We use the following theorem to prove the existence of maximal integral curves (Theorem 2.6).

**Theorem 1 (Picard-Lindelöf)** *Let* $I \subset \mathbf{R}$ *and* $U \subset \mathbf{R}^{n+1}$. *Suppose that*

$$\mathbf{F}\colon I \times U \to \mathbf{R}^{n+1}$$

*is continuous in the first variable and Lipschitz continuous in the second. Fix* $t_0 \in I$ *and* $\mathbf{x}_0 \in U$. *There exists an* $\varepsilon > 0$ *and a unique function* $\mathbf{x}$ *defined on* $I \cap (t_0 - \varepsilon, t_0 + \varepsilon)$ *with values in* $\mathbf{R}^{n+1}$ *such that*

$$\mathbf{x}'(t) = \mathbf{F}(t, \mathbf{x}(t))$$

*and* $\mathbf{x}(t_0) = \mathbf{x}_0$.

A proof of this was published by Lindelöf in 1894 for $n = 0$, but the same proof applies for any $n \in \mathbf{N}$. A copy of the paper is available on the BNF website:

http://gallica.bnf.fr/ark:/12148/bpt6k3074r/f454.table

I will also post a copy on the course website. In addition to this, the statement and proof will be in many stardard text books, for example W. Hurewicz, *Lectures on Ordinary Differential Equations* (1958) p. 28. There is a Wikipedia page, which summarises the proof in English:

http://en.wikipedia.org/wiki/Picard-Lindelöf_theorem