(1) (a) Given a smooth vector field $\mathbf{X}$ on an open set $U \subseteq \mathbf{R}^{n+1}$, define the notion of an integral curve of $\mathbf{X}$. **[3 marks]**

(b) State precisely a theorem regarding the existence and uniqueness of maximal integral curves of a smooth vector field $\mathbf{X}$ on an open set $U \subseteq \mathbf{R}^{n+1}$ through a point $p \in U$. **[7 marks]**

(c) A vector field $\mathbf{X}$ on $\mathbf{R}^2$ is defined by $\mathbf{X}(q) = (q, -q/3)$ for all $q \in \mathbf{R}^2$.

(i) Sketch the vector field $\mathbf{X}$. **[2 marks]**

(ii) Show that finding an integral curve $\alpha\colon I \to \mathbf{R}^2$ of $\mathbf{X}$ through $p \in \mathbf{R}^2$ is equivalent to solving the first order system

$$\begin{cases} x'(t) = -x(t)/3 \\ y'(t) = -y(t)/3 \end{cases}$$

subject to the initial conditions $x(0) = p_1$ and $y(0) = p_2$, where $p = (p_1, p_2)$. **[3 marks]**

(iii) Either by solving the system above, or by some other method, find the maximal integral curve of $\mathbf{X}$ through p for all $p \in \mathbf{R}^2$. **[3 marks]**

(iv) Is the vector field $\mathbf{X}$ complete? Justify your answer. **[2 marks]**

(2) (a) Define what it means for S to be an n-surface in $\mathbf{R}^{n+1}$. **[4 marks]**

(b) For $p \in S$, state the definition of S_p, the tangent space of S at p. **[4 marks]**

(c) The set $\mathbf{R}^4$ may be viewed as the set of all 2×2 matrices with real entries by identifying the quadruple (x_1, x_2, x_3, x_4) with the matrix

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

The subset consisting of those matrices with determinant equal to one forms a group under matrix multiplication, this group is called the special linear group $SL(2)$.

(i) Show that $SL(2)$ is a 3-surface in $\mathbf{R}^4$. [Hint: The determinant of the matrix above is $x_1x_4 - x_2x_3$.] **[5 marks]**

(ii) The trace of a matrix

$$A = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

is defined to be $\mathrm{tr}(A) := x_1 + x_4$. Show that the tangent space $SL(2)_p$ to $SL(2)$ at $p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ can be identified with the set of all 2×2 matrices of trace zero by showing that

$$SL(2)_p = \left\{ \left(p, \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \right) \,\middle|\, x_1 + x_4 = 0 \right\}.$$

[7 marks]

[Please turn over]

(3) (a) Define the Gauss map for an n-surface S with orientation $\mathbf{n}$. **[2 marks]**

(b) Sketch the image of the Gauss map N for the 1-surface $f^{-1}(0)$ with orientation $\nabla f/\|\nabla f\|$ when f is given as follows.

(i) $f(x_1, x_2) = x_2 - x_1^2$ for all $(x_1, x_2) \in \mathbf{R}^2$, and **[2 marks]**

(ii) $f(x_1, x_2) = x_1$ for all $(x_1, x_2) \in \mathbf{R}^2$. **[2 marks]**

(c) State precisely a theorem regarding the Gauss map that gives a condition under which it is surjective. **[5 marks]**

(d) Let $S = f^{-1}(0)$ be an oriented n-surface with orientation

$$\mathbf{n} = (\cdot, N(\cdot)) = \nabla f/\|\nabla f\|$$

for some smooth $f\colon \mathbf{R}^{n+1} \to \mathbf{R}$, let $p_1, p_2 \in S$ and $q \in \mathbf{S}^n$. Suppose that there exists a continuous function $\alpha\colon [a, b] \to \mathbf{R}^{n+1}$, differentiable at a and b, such that

(i) $\alpha(a) = p_1$, $\alpha(b) = p_2$, $\dot{\alpha}(a) = (p_1, q)$ and $\dot{\alpha}(b) = (p_2, q)$, and

(ii) $\alpha(t) \notin S$ for $a < t < b$.

Prove that $N(p_1) \neq N(p_2)$. [Hint: Consider $f \circ \alpha$.] **[9 marks]**

(4) (a) Let S be an oriented n-surface with orientation $\mathbf{n}$. Define the Weingarten map L_p for a $p \in S$. **[5 marks]**

(b) State precisely a theorem which relates the value of the Weingarten map $L_p(\mathbf{v})$ at $\mathbf{v} \in S_p$ and the acceleration of a parametrised curve $\alpha\colon I \to S$ with velocity $\dot{\alpha}(t_0) = \mathbf{v}$. **[7 marks]**

(c) Show that all integral curves of a smooth tangent vector field $\mathbf{X}$ on S are geodesics if and only if $\nabla_{\mathbf{X}(p)}\mathbf{X}(p) \perp S_p$ for all $p \in S$. **[8 marks]**

(5) The length $\ell(\alpha)$ of a parametrised curve $\alpha\colon (a, b) \to \mathbf{R}^{n+1}$ is defined to be

$$\ell(\alpha) = \int_a^b \|\dot{\alpha}(t)\| dt,$$

where $-\infty < a < b < \infty$.

(a) For $-\infty < c < d < \infty$, suppose that $\beta = \alpha \circ h$, where $h\colon [c, d] \to [a, b]$ is a continuous function such that $h'(t) > 0$ for all $t \in (c, d)$, $h(c) = a$ and $h(d) = b$. Show that $\ell(\alpha) = \ell(\beta)$. **[6 marks]**

(b) State precisely a theorem which gives a dichotomy of unit speed global parametrisations of connected oriented plane curves. Which alternative occurs for compact plane curves? **[5 marks]**

(c) Let C be a connected oriented plane curve with orientation $\mathbf{n}$, let $\alpha\colon I \to C$ be a one-to-one unit speed parametrisation of C and let $\kappa\colon C \to \mathbf{R}$ denote curvature on C. Show that

$$(\kappa \circ \alpha)(t) = -\dot{\alpha} \cdot (\mathbf{n} \circ \alpha)^{\cdot}(t)$$

and use this to show that

$$\int_a^b |(\kappa \circ \alpha)(t)| dt = \ell(N \circ \alpha),$$

where a and b are the end-points of I, and $N\colon C \to \mathbf{S}^1 \subset \mathbf{R}^2$ is the Gauss map of C. **[9 marks]**

[End of Paper]