Fourier Analysis
Second Semester 2008/9
Homework Assignment 5
(Due on 20th March 2009)

Only the questions marked with an asterisk (*) will count towards the assessment for this course. Most of these exercises are taken from Stein and Shakarchi.

*1. The aim of this question is to prove the *Heisenberg uncertainty principle*: Suppose ψ is a Schwartz function such that
$$\int_{-\infty}^{\infty} |\psi(x)|^2\,dx = 1.$$
Then
$$\left(\int_{-\infty}^{\infty} x^2|\psi(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty} \xi^2|\widehat{\psi}(\xi)|^2\,d\xi\right) \geq \frac{1}{16\pi^2}. \qquad (\dagger)$$
(a) Show that, under the hypothesis on ψ,
$$1 = -\int_{-\infty}^{\infty}\left(x\psi'(x)\overline{\psi(x)} + x\overline{\psi'(x)}\psi(x)\right)\,dx. \qquad (\ddagger)$$
(b) Use ($\ddagger$) to conclude
$$1 \leq 2\left(\int_{-\infty}^{\infty} x^2|\psi(x)|^2\,dx\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty} |\psi'(x)|^2\,dx\right)^{\frac{1}{2}}$$
(c) Use the above and what you know about the Fourier transform to complete the proof of ($\dagger$).

*2. Define the *Landau kernels* $L_n\colon \mathbf{R} \to \mathbf{C}$ by
$$L_n(x) = \begin{cases} \frac{(1-x^2)^n}{c_n}, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| > 1. \end{cases}$$
where c_n is chosen so that $\int_{\mathbf{R}} L_n = 1$.
(a) Prove that $\{L_n\}_{n\in\mathbf{N}}$ is a family of good kernels as $n \to \infty$.
(b) Prove that, for a bounded uniformly continuous function f, the convolution $(f*L_n)(x) \to f(x)$ uniformly in x.
(c) Show that given a (uniformly) continuous function $f\colon \mathbf{R} \to \mathbf{C}$ which is zero outside $[-1/2, 1/2]$ and $\varepsilon > 0$, we can find a polynomial P such that $|f(x) - P(x)| < \varepsilon$ for all $x \in [-1/2, 1/2]$.

*3 Recall the heat kernel H_t is defined as
$$H_t(x) = \frac{1}{(4\pi t)^{\frac{1}{2}}}e^{-x^2/(4t)}.$$
Show that the function
$$v(x,t) = \frac{x}{t}H_t(x)$$
satisfies the heat equation for $t > 0$ and $\lim_{t\to 0} v(x,t) = 0$ for all $x \neq 0$, but v is not continuous at the origin.

4. For $f \in \mathcal{S}(\mathbf{R})$, suppose $f(x)$ is non-negative for all $x \in \mathbf{R}$, positive for $|x| \leq 1$ and zero for $|x| \geq 2$. Show that $u(x,t) > 0$ for all $x \in \mathbf{R}$ and $t > 0$. (This means that the heat equation propagates heat at infinite speed: here our initial heat distribution f was only non-zero in a neighbourhood of the origin, but for any positive t that heat is felt everywhere on $\mathbf{R}$.)

5. Consider the operators A and A^* defined on $f \in \mathcal{S}(\mathbf{R})$ by
$$Af(x) = f'(x) + xf(x) \quad\text{and}\quad A^*f(x) = -f'(x) + xf(x).$$
The operators A and A^* are sometimes called the *annihilation* and *creation* operators, respectively. Prove that for all $f, g \in \mathcal{S}(\mathbf{R})$ and
$$(f,g) = \int_{\mathbf{R}} f(x)\overline{g(x)}\,dx$$
we have
(a) $(Af, g) = (f, A^*g)$,
(b) $(A^*Af, g) \geq 0$,
(c) $A^*A = L - I$, where $L(f)(x) = -f''(x) + x^2f(x)$ and I is the identity operator.
Use these facts to prove $L(f) \geq f$ for all $f \in \mathcal{S}(\mathbf{R})$ and show this is equivalent to the Heisenberg uncertainty principle of question 1.