Fourier Analysis
Second Semester 2008/9
Homework Assignment 1
(Due on 23rd January 2009)

1. For $z \in \mathbf{C}$, we define the *complex exponential* by
$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
   (a) Show that the above definition makes sense by showing that the series converges for every complex number z.
   (b) If z_1 and z_2 are two complex numbers, prove that $e^{z_1}e^{z_2} = e^{z_1+z_2}$. [Hint: You may like to use the binomial theorem.]
   (c) Show that
$$e^{iy} = \cos y + i \sin y$$
   whenever $y \in \mathbf{R}$. This is called Euler's identity. [Hint: Take sine and cosine to also be defined via their power series.]
   (d) More generally, prove that
$$e^z = e^x(\cos y + i \sin y),$$
   where $z = x + iy$ and $x, y \in \mathbf{R}$.
2. Verify that the function $x \mapsto e^{inx}$ is periodic with period 2π and that
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\,dx = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$
   Use this fact to prove that if $n, m \geq 1$ we have
$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \cos mx\,dx = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m. \end{cases}$$
   and similarly
$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \sin mx\,dx = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m. \end{cases}$$
   Finally, show that
$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \sin mx\,dx = 0$$
   for any n and m. [Hint: Calculate $e^{inx}e^{-imx} + e^{inx}e^{imx}$ and $e^{inx}e^{-imx} - e^{inx}e^{imx}$.]
3. In lectures we arrived at the differential equation
$$y''(t) + c^2 y(t) = 0 \qquad (\dagger)$$
   as a model for the displacement $y(t)$ of a mass attached to a spring as time t passes.
   (a) Show that the function
$$y(t) = a\cos(ct) + b\sin(ct) \qquad (\ddagger)$$
   solves (†), where $a, b \in \mathbf{C}$.
   (b) Show that any solution y of (†) which is twice continuously differentiable is of the form (‡). [Hint: Start by differentiating the two functions $g(t) = y(t)\cos(ct) - c^{-1}y'(t)\sin(ct)$ and $h(t) = y(t)\sin(ct) - c^{-1}y'(t)\cos(ct)$.]
4. Suppose F is a function on (a, b) with two continuous derivatives. Show that whenever x and $x + h$ belong to (a, b), one may write
$$F(x+h) = F(x) + hF'(x) + \frac{h^2}{2}F''(x) + h^2\varphi_x(h)$$
   where $\varphi_x(h) \to 0$ as $h \to 0$ for each x. Deduce that, for each x,
$$\frac{F(x+h) + F(x-h) - 2F(x)}{h^2} \to F''(x)$$
   as $h \to 0$.

5. In lectures we have defined the Fourier sine cofficients of a function f as

$$A_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx$$

for $n = 1, 2, 3, \ldots$. Show that

$$A_n = \frac{2h}{m^2}\frac{\sin(np)}{p(\pi - p)}$$

when $f\colon [0,\pi] \to \mathbf{C}$ is defined as

$$f(x) = \begin{cases} \frac{xh}{p}, & \text{if } 0 \le x \le p, \\ \frac{h(\pi - x)}{\pi - p} & \text{if } p \le x \le x. \end{cases}$$

Observe, this function f may be interpreted as the initial position of a plucked string, and so is a relevant example given our physical motivation.

6. This excercise will show how the symmetries of a function imply certain properties of Fourier coefficients. Let f be a 2π-periodic integrable function defined on $\mathbf{R}$.
   (a) Show that the Fourier series of f can be written as

   $$\hat{f}(0) + \sum_{n=1}^{\infty}(\hat{f}(n) + \hat{f}(-n))\cos(nx) + i(\hat{f}(n) - \hat{f}(-n))\sin(nx).$$

   (b) Prove that if f is even, then $\hat{f}(n) = \hat{f}(-n)$, and so the Fourier series is a cosine series.
   (c) Prove that if f is odd, then $\hat{f}(n) = -\hat{f}(-n)$, and so the Fourier series is a sine series.
   (d) Suppose that $f(x+\pi) = f(x)$ for all $x \in \mathbf{R}$. Show that $\hat{f}(n) = 0$ for all odd n.
   (e) Show that f is real-valued if and only if $\overline{\hat{f}(n)} = \hat{f}(-n)$.

These exercises are taken from Stein and Shakarchi.