Fourier Analysis
Second Semester 2009/10
Homework Assignment 2 Solutions

QUESTION 1.

For $\delta \in (0, \pi)$, let f be defined on $[-\pi, \pi]$ by

$$f(\theta) = \begin{cases} 0, & \text{if } |\theta| > \delta, \\ 1 - |\theta|/\delta, & \text{if } |\theta| \leq \delta. \end{cases}$$

(a) Plot the graph of f.
(b) Show that

$$f(\theta) = \frac{\delta}{2\pi} + 2\sum_{n=1}^{\infty} \frac{1 - \cos(n\delta)}{n^2\pi\delta} \cos(n\theta).$$

Solution.

(a) This is straightforward.
(b) We begin by computing the Fourier coefficients of f. For each $n \neq 0$ these are

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta = \frac{1}{2\pi}\int_{-\delta}^{\delta}(1 - |\theta|/\delta)e^{-in\theta}d\theta \\ &= \frac{1}{\pi}\int_0^{\delta}(1 - \theta/\delta)\cos(n\theta)d\theta = \frac{1}{n\pi\delta}\int_0^{\delta}\sin(n\theta)d\theta \\ &= \frac{1}{n^2\pi\delta}(1 - \cos(n\delta)),\end{aligned}$$

where we used the fact that $e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$, f and cosine are even functions, and sine is odd. The zeroth Fourier coefficient is

$$\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)dx = \frac{\delta}{2\pi}.$$

Clearly f is continuous (as a 2π-periodic function) and the previous calculation allows us to conclude that

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| \leq C \sum_{n=-\infty}^{\infty} \frac{1}{n^2} < \infty.$$

Consequently, by Corollory 2.3.1,

$$\lim_{N\to\infty} S_N(f)(\theta) = f(\theta) \tag{1}$$

for all $\theta \in [-\pi, \pi]$, where

$$S_N(f)(\theta) = \sum_{n=-N}^{N} \hat{f}(n)e^{in\theta}.$$

Once again using the fact that sine is odd, we see that for each $\theta \in [-\pi, \pi]$

$$\begin{aligned}S_N(f)(x) &= \sum_{n=-N}^{-1} \frac{(1 - \cos(n\delta))}{n^2\pi\delta}e^{in\theta} + \frac{\delta}{2\pi} + \sum_{n=1}^{N} \frac{(1 - \cos(n\delta))}{n^2\pi\delta}e^{in\theta} \\ &= \frac{\delta}{2\pi} + 2\sum_{n=1}^{N} \frac{(1 - \cos(n\delta))}{n^2\pi\delta} \cos(n\theta).\end{aligned} \tag{2}$$

Therefore, combining (1) and (2), we find that

$$\begin{aligned}f(\theta) &= \lim_{N\to\infty}\left(\frac{\delta}{2\pi} + \sum_{n=1}^{N} \frac{(1 - \cos(n\delta))}{n^2\pi\delta} \cos(n\theta)\right) \\ &= \frac{\delta}{2\pi} + 2\sum_{n=1}^{\infty} \frac{1 - \cos(n\delta)}{n^2\pi\delta} \cos(n\theta),\end{aligned}$$

as required.

Mark Scheme.

(a) Correct shape, 1pt; correct intersection with axes, 1pt.
(b) Correctly calculate Fourier coefficients, 2pts; observe the Fourier series is absolutely convergent and f is continuous, 3pts; Apply Corollary 2.3.1, 3pts. Other correct and complete method, 8pts.

Notes.

1. A series $\sum_{n=-\infty}^{\infty} a_n$ is said to converge absolutely if there exists a $C > 0$ such that $\sum_{n=-N}^{N} |a_n| \leq C$ for all $N \in \mathbf{N}$. It only makes sense to consider the absolute convergence of a function when it is defined as a series. This is the case for the Fourier series of a function, although be aware that a priori there is no reason to think a Fourier series of an integrable function defines a function at all. It does not make sense to say the function f in the present question is absolutely convergent, as it is not defined via a series.

   Observe that requiring that there exists $C > 0$ such that $\left|\sum_{n=-N}^{N} a_n\right| \leq C$ for all $N \in \mathbf{N}$ is a strictly weaker condition than absolute convergence (consider the example $a_n = (-1)^n$ for all $n \in \mathbf{Z}$).

   You will recall from your first year that requiring $a_n \to 0$ as $n \to \infty$ does not even guarantee pointwise convergence of the series $\sum_{n=-\infty}^{\infty} a_n$ (consider the example $a_n = 1/|n|$ for $n \neq 0$).
2. Other methods may also be used to answer this question. Any correct method can be used to answer a question unless the question asks that a particular method be used, however, if you choose to use results not covered during your degree, I expect you to have a complete understanding of their proof. All questions should be answerable with the results you have seen in class and it is my intention that questions test the material covered in class. If you quote a result not covered in class without proof, I may later ask you to demonstrate your understanding of the proof.

   One such method attempted by some students was to use a theorem called Jordan's Criterion. To apply this result successfully you must first quote a source for the proof of the theorem, or better, prove the theorem yourself in your written answer. Next, as when applying any theorem, you must check the conditions of the theorem hold in the case you want to apply it.

   Another student attempted to apply a result which held for differentiable functions. Unfortunately, f is not differentiable everywhere.

   A third alternative method attempted was to use Carleson's result that the Fourier series of an $L^2(\mathbf{T})$ function converges almost everywhere — almost everywhere here has a very specific meaning. However, obviously this only gives the required equality for almost every θ, not all $\theta \in [-\pi, \pi]$. Moreover, the proof of Carleson's result is very hard. If you happen to have read and understood the proof of this result, I would be very interested in spending some time discussing it with you.

Question 7

The aim of this question is to evaluate the improper Riemann integral

$$\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{b\to\infty} \int_0^b \frac{\sin x}{x}\,dx.$$

(a) Show that

$$\lim_{b\to\infty} \int_0^b \frac{\sin x}{x}\,dx = \lim_{N\to\infty} \int_0^\pi \frac{\sin((N+1/2)x)}{x}\,dx.$$

(b) Use the Riemann-Lebesgue Lemma (as stated in question 6) to prove

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \left(\frac{1}{\sin(x/2)} - \frac{2}{x}\right) \sin((N+1/2)x)\,dx = 0.$$

(c) Use the above and the fact that

$$\int_{-\pi}^{\pi} \frac{\sin((N+1/2)x)}{\sin(x/2)}\,dx = 2\pi$$

for all $N \geq 1$ to show that $\int_0^\infty \frac{\sin x}{x}\,dx = \pi/2$.

Solution.

(a) First we apply the change of variables $y(N(b)+1/2)=x$, where $N(b)$ will be chosen below depending only on b, to obtain

$$\begin{aligned}\int_0^b \frac{\sin x}{x}\,dx &= \int_0^{b/(N(b)+1/2)} \frac{\sin((N(b)+1/2)y)}{y}\,dy \\ &= \int_0^{\pi} \frac{\sin((N(b)+1/2)y)}{y}\,dy - \int_{b/(N(b)+1/2)}^{\pi} \frac{\sin((N(b)+1/2)y)}{y}\,dy.\end{aligned} \tag{3}$$

For a given $b>0$ choose $N(b)$ to be the unique integer such that

$$\frac{b}{\pi}-\frac{1}{2} < N(b) \le \frac{b}{\pi}+\frac{1}{2}.$$

Observe then that

$$\frac{\pi}{1+\pi/b} \le \frac{b}{(N(b)+1/2)} < \pi,$$

so

$$\lim_{b\to\infty} \frac{b}{(N(b)+1/2)} = \pi$$

by the Sandwich Principle. Therefore

$$\left|\int_{b/(N(b)+1/2)}^{\pi} \frac{\sin((N(b)+1/2)y)}{y}\,dy\right| \le \int_{b/(N(b)+1/2)}^{\pi} \frac{1}{y}\,dy \to 0$$

as $b\to\infty$ because $y\mapsto 1/y$ is bounded near $y=\pi$. From this and (3) we see that

$$\lim_{b\to\infty}\int_0^b \frac{\sin x}{x}\,dx = \lim_{b\to\infty}\int_0^{\pi} \frac{\sin((N(b)+1/2)y)}{y}\,dy.$$

Finally we notice that $b\mapsto N(b)$ is a surjective increasing function from positive real numbers to the non-negative natural numbers, consequently the right-hand limit above can be replaced with a limit over the natural numbers:

$$\lim_{b\to\infty}\int_0^b \frac{\sin x}{x}\,dx = \lim_{N\to\infty}\int_0^{\pi} \frac{\sin((N+1/2)y)}{y}\,dy.$$

(b) We have that

$$\begin{aligned}\int_{-\pi}^{\pi}\left(\frac{1}{\sin(x/2)}-\frac{2}{x}\right)\sin((N+1/2)x)\,dx &= \operatorname{Im}\left(\int_{-\pi}^{\pi}\left(\frac{1}{\sin(x/2)}-\frac{2}{x}\right)e^{i(N+1/2)x}\,dx\right) \\ &= \operatorname{Im}\left(\int_{-\pi}^{\pi}\left(\frac{1}{\sin(x/2)}-\frac{2}{x}\right)e^{ix/2}e^{iNx}\,dx\right) \\ &= \operatorname{Im}\left(2\pi\hat{f}(-N)\right),\end{aligned}$$

where

$$f(x)=\left(\frac{1}{\sin(x/2)}-\frac{2}{x}\right)e^{ix/2}.$$

Thus, we would obtain the desired result if we could applying the Riemann-Lebesgue Lemma to the function f. To do this we must check that f is Riemann integrable. We know that continuous functions are integrable, so it suffices to check that f is continuous. Certainly $x\mapsto e^{ix/2}$ is continuous and the product of continuous functions in continuous, so it suffices to check $x\mapsto(\frac{1}{\sin(x/2)}-\frac{2}{x})$ is continuous. This is clearly the case for $x\neq 0$, but we must consider the behaviour near zero more closely. We have that

$$\lim_{x\to 0}\left(\frac{1}{\sin(x/2)}-\frac{2}{x}\right) = \lim_{x\to 0}\left(\frac{x-2\sin(x/2)}{x\sin(x/2)}\right),$$

and applying L'Hôpital's Rule twice to the limit on the right-hand side shows that it exists (and equals zero). Consequently f is continuous everywhere (if we define $f(0)=0$) and so is integrable. Thus, by the Riemann-Lebesgue Lemma,

$$\begin{aligned}&\lim_{N\to\infty}\int_{-\pi}^{\pi}\left(\frac{1}{\sin(x/2)}-\frac{2}{x}\right)\sin((N+1/2)x)\,dx \\ &= \lim_{N\to\infty}\operatorname{Im}\left(2\pi\hat{f}(-N)\right) = \operatorname{Im}\left(2\pi\lim_{N\to\infty}\hat{f}(-N)\right) = 0.\end{aligned}$$

(c) We have that

$$\begin{aligned}
2\pi &= \lim_{N\to\infty} \int_{-\pi}^{\pi} \frac{\sin((N+1/2)x)}{\sin(x/2)}\,dx \\
&= \lim_{N\to\infty} \int_{-\pi}^{\pi} \left(\frac{1}{\sin(x/2)} - \frac{2}{x}\right) \sin((N+1/2)x)\,dx + \lim_{N\to\infty} \int_{-\pi}^{\pi} \frac{2\sin((N+1/2)x)}{x}\,dx \\
&= 0 + \lim_{N\to\infty} 2 \int_{-\pi}^{\pi} \frac{\sin((N+1/2)x)}{x}\,dx \\
&= \lim_{N\to\infty} 4 \int_{0}^{\pi} \frac{\sin((N+1/2)x)}{x}\,dx \\
&= \lim_{b\to\infty} 4 \int_{0}^{b} \frac{\sin x}{x}\,dx = 4 \int_{0}^{\infty} \frac{\sin x}{x}\,dx.
\end{aligned}$$

Mark Scheme.

(a) Perform a correct substitution, 1pt; take into account the fact that one limit is over $\mathbf{R}$ and the other is over $\mathbf{N}$ (if not in this part of the question, then else where in the answer), 1pt.

(b) Correctly rearrange integral so it is in a form where the Riemann-Lebesgue Lemma can be used or adapt the statement of the Riemann-Lebesgue Lemma appropriately, 2pts; find a valid function to apply this result, 1pt; check the conditions of the Lemma, 2pts.

(c) Put together the three steps, 3pts.

Notes.

1. The functions $x \mapsto 1/x$ and $x \mapsto 1/\sin(x)$ are not integrable near zero, no matter what you might read on the internet. Note this is a different statement to saying they have primatives near zero. If you quote a source other than your notes for this course or earlier courses, you should be sure that you understand the proof of any statement you use.
2. The quantity $0/0$ is not defined, and so no limit can be equal to $0/0$ — even the limit of a quotient when the limit of the denominator and numerator are both zero.