Fourier Analysis
Second Semester 2009/10
Homework Assignment 4
(Due on 16th March 2010. Please staple multiple sheets together.)

Only the questions marked with an asterisk (*) will count towards the assessment for this course.

*1. In lectures we have proved the Heisenberg uncertainty principle, which says for $\psi \in \mathcal{S}(\mathbf{R})$ such that $\|\psi\|_{L^2(\mathbf{R})} = 1$ we have that

$$\left(\int_{-\infty}^{\infty} x^2|\psi(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty} \xi^2|\widehat{\psi}(\xi)|^2\,d\xi\right) \geq \frac{1}{16\pi^2}.$$

Use this to prove that

$$\left(\int_{-\infty}^{\infty} (x-x_0)^2|\psi(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty} (\xi-\xi_0)^2|\widehat{\psi}(\xi)|^2\,d\xi\right) \geq \frac{1}{16\pi^2}$$

for each $x_0, \xi_0 \in \mathbf{R}$.

2. Define the *Landau kernels* $L_n\colon \mathbf{R} \to \mathbf{C}$ by

$$L_n(x) = \begin{cases} \frac{(1-x^2)^n}{c_n}, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| > 1. \end{cases}$$

where c_n is chosen so that $\int_{\mathbf{R}} L_n = 1$.

(a) Prove that $\{L_n\}_{n\in\mathbf{N}}$ is a family of good kernels as $n \to \infty$.
(b) Prove that, for a bounded uniformly continuous function f, the convolution $(f*L_n)(x) \to f(x)$ uniformly in x.
(c) Show that given a (uniformly) continuous function $f\colon \mathbf{R} \to \mathbf{C}$ which is zero outside $[-1/2, 1/2]$ and $\varepsilon > 0$, we can find a polynomial P such that $|f(x) - P(x)| < \varepsilon$ for all $x \in [-1/2, 1/2]$.

3 Consider the initial value problem of finding $u\colon \mathbf{R} \times [0,\infty) \to \mathbf{C}$ such that

$$\begin{cases} \partial_t u(x,t) - \partial_x^2 u(x,t) = 0 & \text{for } x \in \mathbf{R} \text{ and } t > 0, \\ u(x,0) = f(x) & \text{for } x \in \mathbf{R}, \end{cases}$$

given data $f \in \mathcal{S}(\mathbf{R})$.

(a) Use the Fourier transform to derive a formula for a solution to be

$$u(x,t) = (H_t * f)(x) \qquad (\star)$$

where H_t is the heat kernel and is defined as

$$H_t(x) = \frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-x^2/(4t)},$$

and $*$ denotes convolution in $\mathbf{R}$.

(b) Show directly that the Heat kernel satisfies

$$\partial_t H_t(x) - \partial_x^2 H_t(x) = 0$$

for $x \in \mathbf{R}$ and $t > 0$.

(c) Show that $\{H_t\}_{t>0}$ is a family of good kernels and so $\|u(\cdot,t) - f\|_{L^2(\mathbf{R})} \to 0$ as $t \to 0$.
(d) Show that the function

$$v(x,t) = \frac{x}{t} H_t(x)$$

satisfies the heat equation for $t > 0$ and $\lim_{t\to 0} v(x,t) = 0$ for all $x \neq 0$, but v is not continuous at the origin.

4. For $f \in \mathcal{S}(\mathbf{R})$, suppose $f(x)$ is non-negative for all $x \in \mathbf{R}$, positive for $|x| \leq 1$ and zero for $|x| \geq 2$. Show that $u(x,t) > 0$ for all $x \in \mathbf{R}$ and $t > 0$, for u is given by formula $(\star)$. (This means that the heat equation propagates heat at infinite speed: here our initial heat distribution f was only non-zero in a neighbourhood of the origin, but for any positive t that heat is felt everywhere on $\mathbf{R}$.)

*5. For each $t \in \mathbf{R}$, consider the operators A_t and A_t^* defined on $f \in \mathcal{S}(\mathbf{R})$ by

$$A_t f(x) = f'(x) + txf(x) \quad \text{and} \quad A_t^* f(x) = -f'(x) + txf(x).$$

When $t = 1$ the operators A_t and A_t^* are sometimes called the *annihilation* and *creation* operators, respectively. Prove that for all $f, g \in \mathcal{S}(\mathbf{R})$ and

$$(f,g) = \int_{\mathbf{R}} f(x)\overline{g(x)}\,dx$$

we have

(a) $(A_t^* f, g) = (f, A_t g)$,

(b) $(A_t^* A_t f, g) \geq 0$,

(c) $A_t^* A_t = L_t - tI$, where $L_t(f)(x) = -f''(x) + t^2 x^2 f(x)$ and I is the identity operator.

Use these facts to prove $(L_t(f), f) \geq t(f, f)$ for all $f \in \mathcal{S}(\mathbf{R})$ and use it to provide an alternative proof of the Heisenberg uncertainty principle.

6. Suppose $g \colon \mathbf{R} \times [0,1] \to \mathbf{C}$ is such that $g(\cdot, t) \in \mathcal{M}(\mathbf{R})$ and $\partial_2 g(\cdot, t) \in \mathcal{M}(\mathbf{R})$ for each $t \in [0,1]$ and there exists a function $\Phi \in \mathcal{M}(\mathbf{R})$ such that

$$|\partial_2 g(x,t) - \partial_2 g(x,s)| \leq |t-s|\Phi(x)$$

for all $s, t, x \in \mathbf{R}$. Here ∂_2 denotes differentiation with respect to the second variable. Prove that $G(\cdot) = \int_{-\infty}^{\infty} g(x, \cdot)dx$ is a differentiable function and

$$G'(t) = \int_{-\infty}^{\infty} \partial_2 g(x,t)dx.$$