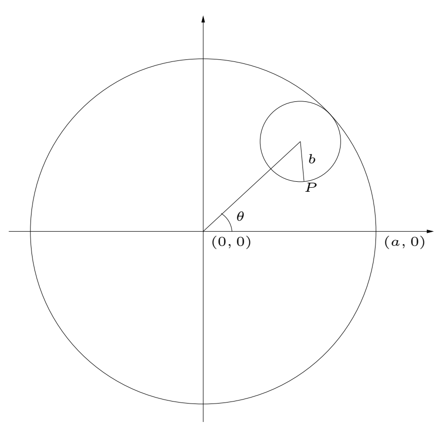
Mathematical Methods for Social Scientists
Math 195 (Sec 55), Autumn 2006
Revision Sheet for Mid-term 1

This mid-term will cover those sections in the text book which we have studied in class. From Chapter 11 we studied parametric equations, sketching a parametric curve and calculus with parametric curves. We studied most of the sections in chapter 13, and vector functions, and their derivatives and integrals in chapter 14. The following questions are of the style you can expect in the exam.

1. A cycloid is the curve γ in the xy-plane swept out by a point on the edge of a wheel as the wheel rolls along the x-axis.
   (a) Draw a diagram of the wheel after it has turned an angle θ to help explain how the cycloid is obtained. Label the point on the edge of the wheel P, the centre of the wheel C and the point where P was when the wheel began to turn O.
   (b) Use the diagram to derive a parametric equation for the cyloid γ, parametrised by the angle θ.
   (c) Calculate dy/dx for this curve using your parametrisation. Where is dy/dx *not* defined?

2. Derive the parametric equations of a *hypercyloid*. This is the curve traced out by a point $P(x, y)$ fixed on a wheel or raduis b as it rolls around the inside of a larger wheel of radius a centred at the origin. Parametrise the curve using the angle θ between the x-axis and the ray between the origin and the centre of the smaller wheel.



3. (a) Write down the expression that gives the arc length of a curve γ in $\mathbf{R}^2$ given by the parametric equations

$$x = f(t) \quad \text{and} \quad y = g(t),$$

for $t \in [t_1, t_2]$.

(b) Suppose $f(t) = \cos t + t\sin t$, $g(t) = \sin t - t\cos t$, $t_1 = 0$ and $t_2 = 2\pi$. Calculate the arc length of the curve.

(c)

4. The curvature of a curve in the xy-plane given by $y = f(x)$ is defined to be

$$\kappa = \left|\frac{d\phi}{ds}\right|,$$

where $\phi = \arctan(dy/dx)$ is the angle of inclination of the tangent line from the horizontal and s is arc-length. Prove that

$$\kappa = \frac{|f''|}{\left(1 + (f')^2\right)^{\frac{3}{2}}}.$$

5. (a) Write down the equation for a sphere of radius r and centre $\mathbf{r}_0 = (x, y, z)$.

   (b) Consider the points $P(x, y, z)$ such that the distance from P to $A(-1, 5, 3)$ is twice the distance from P to $B(6, 2, -2)$. Show that the set of all such points is a sphere, and find its centre and radius.

   (c) Find an equation for the set of points equidistant from the points $A(-1, 5, 3)$ and $B(6, 2, -2)$. Describe the set.

6. (a) Define the length $|\mathbf{a}|$ of a vector $\mathbf{a} \in V_3$ and the dot product $\mathbf{a} \cdot \mathbf{b}$ of two vectors $\mathbf{a}, \mathbf{b} \in V_3$

   (b) Prove that $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

   (c) Write down an expression relating the dot product of $\mathbf{a}$ and $\mathbf{b}$ and the angle θ between the two vectors. When are two vectors orthogonal?

   (d) Find a $c \in \mathbf{R}$ such that $\langle 3, 4, 8\rangle$ and $\langle -3, 1, c\rangle$ are orthogonal.

7. (a) Define the cross product $\mathbf{a} \times \mathbf{b}$ of two vectors $\mathbf{a}, \mathbf{b} \in V_3$.

   (b) Describe a geometric meaning of $|\mathbf{a} \times \mathbf{b}|$ for two vectors $\mathbf{a}$ and $\mathbf{b}$.

   (c) The equation $2x + 3y + 4z = 12$ is the equation of a plane. Find a normal vector to the plane.

   (d) Find an equation for the plane which passes through the point $(3, 6, -1)$ with normal vector $\langle 3, 2, -1\rangle$.

   (e) Without calculating directly using the components of the vectors, argue that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is zero if any two of the three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are equal.

8. (a) For a vector-valued function $\mathbf{r} \colon \mathbf{R} \to V_3$ with component function $\mathbf{r} = \langle f_1, f_2, f_3\rangle$ define what it means to say

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{a}.$$

   Define the derivative $\mathbf{r}'(t)$, for $t \in \mathbf{R}$.

(b) Calculate a formula for

$$\frac{d}{dt}(f(t)\mathbf{r}(t)).$$

where $\mathbf{r}$ is a vector-valued function and f is a real-valued function.

(c) Calculate a formula for

$$\frac{d}{dt}(\mathbf{r}(t) \times \mathbf{s}(t)).$$

where $\mathbf{r}$ and $\mathbf{s}$ are vector-valued functions.

(d) If $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ and $\mathbf{s}(t) = \langle t, t^2, 1 \rangle$, compute

$$\frac{d}{dt}(\mathbf{r}(t) \cdot \mathbf{s}(t)).$$

(e) With $\mathbf{r}$ as in the previous part of the question compute

$$\int_0^{\frac{\pi}{2}} \mathbf{r}(t)\, dt.$$