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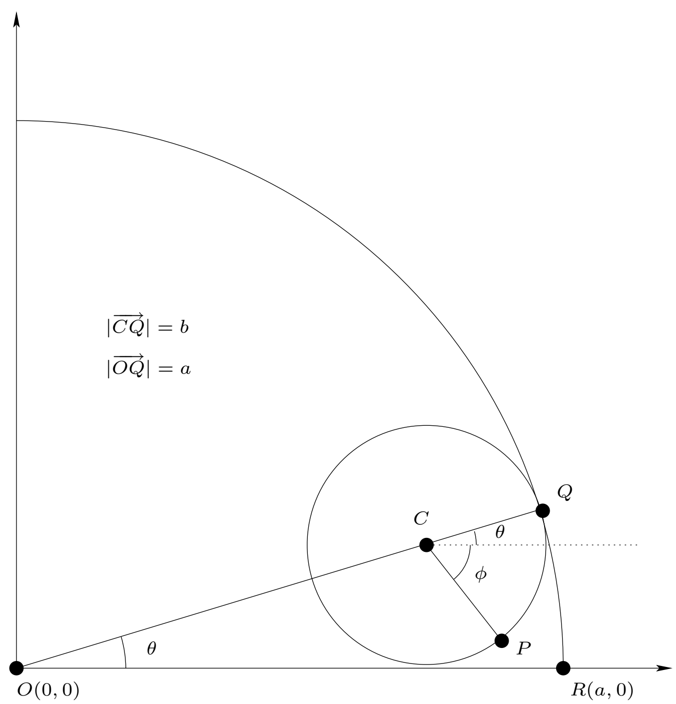
Mathematical Methods for Social Scientists I
Math 195 (Sec 55), Autumn 2006
Mid-term 1
23rd October 2006

Instructions: The total time allowed for this examination is 50 minutes. The use of notes, textbooks or calculators is prohibited. Write you answer to each question in the space provided below it. Should you require more space write on the reverse of the paper, labelling your answers clearly. Write your name at the top of this sheet. The maximum number of points available for each question or part question is shown in parentheses next to the question. There are 5 questions. You should attempt as many of the questions as you can. Partial credit may be given for incomplete answers.

1. A *hypercyloid* is the curve traced out by a point $P(x, y)$ fixed on a wheel of radius b as it rolls around the inside of a larger wheel of radius a centred at the origin. Let θ, ϕ, Q and R be as in diagram below. Suppose P has coordinates $(a, 0)$ when $\theta = 0$.



(5 pts) (a) Use the geometry of the situation to derive an expression relating θ and ϕ.

Since the smaller wheel is rolling around the inside of the larger wheel, we know the arclength from R to Q will equal the arclength from Q to P. Thus $a\theta = b(\theta + \phi)$, and rearranging this we find

$$\phi = \frac{(a-b)}{b}\theta$$

(5 pts) (b) Using this information and anything else you may need, show that the parametric equations for the hypercycloid are

$$x = (a-b)\cos\theta + b\cos\left(\frac{(a-b)\theta}{b}\right) \quad \text{and} \quad y = (a-b)\sin\theta - b\sin\left(\frac{(a-b)\theta}{b}\right).$$

We can compute the coordinates of P by calculating the change in x and y as we move from O to C and then along the radius of the smaller circle to P. Thus

$$x = (a-b)\cos\theta + b\cos(\phi) = (a-b)\cos\theta + b\cos\left(\frac{(a-b)\theta}{b}\right)$$

and

$$y = (a-b)\sin\theta - b\sin(\phi) = (a-b)\sin\theta - b\sin\left(\frac{(a-b)\theta}{b}\right).$$

(5 pts) (c) Using the parametric equations from (b), calculate the slope dy/dx of the curve at $P(x,y)$.

We compute

$$\frac{dx}{d\theta} = -(a-b)\sin\theta - (a-b)\sin\left(\frac{(a-b)\theta}{b}\right)$$

and

$$\frac{dy}{d\theta} = (a-b)\cos\theta - (a-b)\cos\left(\frac{(a-b)\theta}{b}\right).$$

Therefore

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{(a-b)\cos\theta - (a-b)\cos\left(\frac{(a-b)\theta}{b}\right)}{-(a-b)\sin\theta - (a-b)\sin\left(\frac{(a-b)\theta}{b}\right)} = -\frac{\cos\theta - \cos\left(\frac{(a-b)\theta}{b}\right)}{\sin\theta + \sin\left(\frac{(a-b)\theta}{b}\right)}.$$

(5 pts) (d) Show that $dy/dx = 0$ when $\theta = 0$. (Hint: Use L'Hôpital's Rule.)

Although our expression for the derivative is not defined when $\theta = 0$ let's cheat by taking the limit of the gradient as $\theta \to 0$. We have

$$\left.\frac{dy}{dx}\right|_{\theta=0} = -\lim_{\theta\to 0}\frac{\cos\theta - \cos\left(\frac{(a-b)\theta}{b}\right)}{\sin\theta + \sin\left(\frac{(a-b)\theta}{b}\right)} = -\lim_{\theta\to 0}\frac{-\sin\theta + \frac{(a-b)}{b}\sin\left(\frac{(a-b)\theta}{b}\right)}{\cos\theta + \frac{(a-b)}{b}\cos\left(\frac{(a-b)\theta}{b}\right)} = -\frac{0 + \frac{(a-b)0}{b}}{1 + \frac{(a-b)}{b}} = 0.$$

(5 pts) 2. (a) Define the length $|\mathbf{a}|$ of a vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle \in V_3$ and the dot product $\mathbf{a} \cdot \mathbf{b}$ of two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle, \mathbf{b} = \langle b_1, b_2, b_3 \rangle \in V_3$.

The length of a vector $a = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

and the dot product $\mathbf{a} \cdot \mathbf{b}$ of $\mathbf{a}$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

(5 pts) (b) Prove that $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$.

We compute

$$\mathbf{a} \cdot \mathbf{a} = a_1 a_1 + a_2 a_2 + a_3 a_3 = a_1^2 + a_2^2 + a_3^2 = \left(\sqrt{a_1^2 + a_2^2 + a_3^2} \right)^2 = |\mathbf{a}|^2.$$

(5 pts) (c) Write down an expression relating the dot product of $\mathbf{a}$ and $\mathbf{b}$ and the angle θ between the two vectors.

Well, that's just

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta.$$

(5 pts) (d) Find a $c \in \mathbf{R}$ such that $\langle 3, 4, 8 \rangle$ and $\langle -3, 1, c \rangle$ are orthogonal.

Two vectors are orthogonal when their dot product is zero. Therefore we require

$$3 \times (-3) + 4 \times 1 + 8c = 0$$

so $c = 5/8$.

(7 pts) 3. (a) For a vector-valued function $\mathbf{r}\colon \mathbf{R} \to V_3$ with component functions $\mathbf{r} = \langle f_1, f_2, f_3 \rangle$ define what it means to say

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{a}.$$

The limit of a vector-valued function $\mathbf{r}\colon \mathbf{R} \to V_3$ is defined to be

$$\lim_{t \to t_0} \mathbf{r}(t) := \langle \lim_{t \to t_0} f_1(t), \lim_{t \to t_0} f_2(t), \lim_{t \to t_0} f_3(t) \rangle,$$

where the limits on the right-hand side are limits of real-valued functions.

(7 pts) (b) Derive a formula for

$$\frac{d}{dt}(\mathbf{r}(t) \cdot \mathbf{s}(t)).$$

where $\mathbf{r}$ and $\mathbf{s}$ are vector-valued functions.

We will write $\mathbf{s} = \langle g_1, g_2, g_3 \rangle$ and $\mathbf{r} = \langle f_1, f_2, f_3 \rangle$, so then

$$\begin{aligned} \frac{d}{dt}(\mathbf{r}(t) \cdot \mathbf{s}(t)) &= \frac{d}{dt}\left(f_1(t)g_1(t) + f_2(t)g_2(t) + f_3(t)g_3(t)\right) \\ &= f_1'(t)g_1(t) + f_2'(t)g_2(t) + f_3'(t)g_3(t)f_1(t)g_1'(t) + f_2(t)g_2'(t) + f_3(t)g_3'(t) \\ &= \mathbf{r}'(t) \cdot \mathbf{s}(t) + \mathbf{r}(t) \cdot \mathbf{s}'(t). \end{aligned}$$

(6 pts) (c) If $\mathbf{r}(t) = \langle \cos t, e^t \sin t, t^2 \rangle$ and $\mathbf{s}(t) = \langle t, e^t - e^{-t}, 1 \rangle$, compute

$$\frac{d}{dt}(\mathbf{r}(t) \cdot \mathbf{s}(t)).$$

We have $\mathbf{r}'(t) = \langle -\sin t, e^t(\sin t + \cos t), 2t \rangle$ and $\mathbf{s}'(t) = \langle 1, e^t + e^{-t}, 0 \rangle$ so

$$\begin{aligned} &\frac{d}{dt}(\mathbf{r}(t) \cdot \mathbf{s}(t)) = \mathbf{r}'(t) \cdot \mathbf{s}(t) + \mathbf{r}(t) \cdot \mathbf{s}'(t) \\ &= -t \sin t + e^t(\sin t + \cos t)(e^t - e^{-t}) + 2t + \cos t + (e^t + e^{-t})e^t \sin t + 0. \end{aligned}$$

(7 pts) 4. (a) The equation $2x + 3y + 4z = 12$ is the equation of a plane. Find a normal vector to the plane. Find the distance of the plane from the origin.

We can read off the normal vectors from the coefficients of the equation of the plane: the normal is $\mathbf{n} = \langle 2, 3, 4 \rangle$. We know the distance of the origin from the plane is given by the constant term in the Cartesian equation divided by $|\mathbf{n}|$, thus this distance is

$$\frac{12}{|\mathbf{n}|} = \frac{12}{\sqrt{2^2 + 3^2 + 4^2}} = \frac{12}{\sqrt{29}}$$

(7 pts) (b) Find an equation for the plane which passes through the point $(3, 6, -1)$ with normal vector $\langle 3, 2, -1 \rangle$.

The general form of the equation of a plane in $\mathbf{r} = \langle x, y, z \rangle$ is $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$, where $\mathbf{n}$ is the normal and $\mathbf{r}_0$ is a point in the plane. Therefore, the equation of the plane which passes through the point $(3, 6, -1)$ with normal vector $\langle 3, 2, -1 \rangle$ is

$$3x + 2y - z = 3(3) + 2(6) - 1(-1) = 22.$$

(6 pts) (c) Find the Cartesian equation of the plane which contains the line $x = 1 + 2t$, $y = 4 - 5t$, $z = 2 + t$ and the point $(3, 1, 3)$.

To write down the Cartesian equation of a plane we need a normal vector to the plane and a point through which the plane passes. Clearly we want the plane to pass through $P(3, 1, 3)$. To find a normal vector we will find two other points A and B in the plane by plugging in values to t into our equation of the line (which we want to be contained in the plane). Then we can take $\mathbf{n} = \overrightarrow{PA} \times \overrightarrow{PB}$ to be our normal. When $t = 0$ we obtain the point $A(1, 4, 2)$ and when $t = 1$ we obtain the point $B(3, -1, 3)$. Thus $\overrightarrow{PA} = \langle -2, 3, -1 \rangle$ and $\overrightarrow{PB} = \langle 0, -2, 0 \rangle$ so

$$\mathbf{n} = \overrightarrow{PA} \times \overrightarrow{PB} = \langle -2, 3, -1 \rangle \times \langle 0, -2, 0 \rangle = \langle 0 - (-1)(-2), 0 - 0, (-2)(-2) - 0 \rangle = \langle -2, 0, 4 \rangle.$$

Therefore the equation of the plane is

$$-2x + 0y + 4z = -2(3) + 0(1) + 4(3) = 6.$$

(4 pts) 5. **Just for fun:** The following words are acronyms. What do they stand for?

(i) Radar RAdio Detection And Ranging
(ii) Laser Light Amplification by the Stimulated Emission of Radiation
(iii) Scuba Self Contained Underwater Breathing Apparatus
(iv) BMW Bayerische Motoren Werke (Bavarian Motor Works)

Please note that, with the exception of question 5, all my answers are written in sentences. I expect the same from you.