Elementary Functions and Calculus III
Math 133 (Sec 22), Spring 2004
Revision Sheet for the Final Examination

The final will be based on the material we have covered in lectures this quarter. Also remember that material from Maths 131-2 will be assumed, so you should remain familiar with that too. Please don't forget remember I make mistakes too, so if you see something which doesn't look write or I use what you think is new notation, chances are I have made a typo. I will *not* be forgiving if you reproduce mistakes I have made - mathematics should make sense, not least to you. Ask me before the exam if you are unsure.

You should be familiar with the following concepts and be able to recall definitions and major theorems associated to them:

- **Transcendental Functions** This section included a couple of techniques for solving certain differential equations. You should know what these are and which equations they allow you to solve. Be able to apply these techniques to given examples. Recall how we defined the inverse trigonometric functions, and how to differentiate them.
- **Techniques of Integration** "Integration is cheap tricks made systematic" - Jesper Grodal. Try and remember all the possible substitutions. You should also know the integration by parts theorem and it's applications to evaluating certain types of integrals. Don't forget partial fractions.
- **Indeterminate Forms** Recall L'Hôpital's rule in its several forms and all the ways you can use it to evaluate certain limits.
- **Sequences and Series** Recall all the main theorems from this section. Make sure you understand the difference between a sequence and series and if you mean to take a limit use the correct notation - a common mistake in the second mid-term was to write $\frac{1}{n} = 0$, however this is not true for any $n \in \mathbf{N}$ (or even $\mathbf{R}$). Be able to recall Taylor's theorem.

Please remember to read carefully what a question is asking of you. You should always try to justify your steps when they are not immediately obvious, and doing so will convince me you understand the material well, which will make my happy, and so increase your mark.

The following questions are in the style you can expect in the test. It would be good preparation for the exam to try these questions. It is also valuable to check you have answered them correctly.

1. Consider the following differential equation with an initial condition.

$$\begin{cases} y'(x) + (2\cos x)y(x) = \cos x, & \text{for all } x \in \mathbf{R}; \\ y(\pi/6) = 2 \end{cases}$$

   (a) Compute the integrating factor for this first-order linear differential equation.

(b) Use the above or any technique you know to find an expression for y.

2. (a) State the integration by parts theorem.
   (b) Use integration by parts to find $\int_0^{\frac{\pi}{2}} xe^x \, dx$.
   (c) Find an anti-derivative of ln and state the domain where your formula is valid.

3. Use the appropriate trigonometric identity to evaluate the following.
   (a) $\int_{-3}^{3} \sqrt{9 - x^2} \, dx$
   (b) $\int \frac{\sqrt{4-x^2}}{x} \, dx$

4. (a) State L'Hôpital's rule for indeterminate forms of the type $0/0$ and ∞/∞.
   (b) Use L'Hôpital's rule to evaluate the following limits.
      i.
$$\lim_{x \to 0} \frac{x}{1 - \cos x}$$
      ii.
$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$$
      iii.
$$\lim_{x \to 0} \frac{\sin x - x}{x^3}$$

5. (a) Consider a function $f : (-\infty, \infty) \to \mathbf{R}$. Define the improper integral $\int_a^\infty f(x) \, dx$. When do we say such a thing converges or diverges?
   (b) Use the definition above to calculate for which $p \in \mathbf{R}$ $\int_1^\infty x^p \, dx$ converges. What is its value when it converges?
   (c) Define the improper integral $\int_{-\infty}^{\infty} f(x) \, dx$.

6. (a) For a sequence $\{a_n\}_n$ define what it means to say the sequence converges to some limit l. What notation is used to denote this number?
   (b) Prove by using any method you know to show whether or not the sequence $\{a_n\}_n$ converges when a_n is defined as follows for all $n \in \mathbf{N}$. (i) $a_n = \frac{(-1)^n}{n}$, (ii) $a_n = (-1)^n$, (iii) $a_n = (1 + \frac{1}{n})^n$.
   (c) State the alternating series test.

7. (a) Define what we mean by saying $\sum_{n=1}^{\infty} a_n$ converges. What does it mean for this series to diverge?
   (b) State the ratio test for a series. State carefully the required conditions.
   (c) Use the ratio test to prove that $\sum_{n=1}^{\infty} \frac{(-1)^n x^{(2n)}}{(2n)!}$ converges for all $x \in \mathbf{R}$.

Remember, the best preparation the night before the exam is to sleep well. I know this might be hard to do, but a well rested mind thinks more clearly. Good luck.

Answers to the practice problems

1. (a) *Answer.* $\exp(2\sin x)$

   (b) *Answer.*

   $$y(x) = \frac{1}{2}\left(\exp(-(2\sin x)) + 1\right)$$

2. (a) *Answer.* Look in your notes or the text.

   (b) *Answer.* $\{\frac{\pi}{2} - 1\}e^{\frac{\pi}{2}} + 1$

   (c) *Answer.* $A_x(\ln x) = x\ln x - x$ for $x > 0$.

3. (a) *Answer.* $\frac{3\pi}{2}$

   (b) *Answer.* $\arcsin(\frac{x}{2}) + \frac{x}{2}\sqrt{1 - (\frac{x}{2})^2}$

4. (a) *Answer.* Look in your notes or the text.

   (b) i. *Answer.* 1

   ii. *Answer.* $\frac{1}{2}$

   iii. *Answer.* $\frac{-1}{6}$

5. (a) *Answer.* Look in your notes or the text.

   (b) *Answer.* It converges for $p < 1$ and its value is then $\frac{-1}{p+1}$.

   (c) *Answer.* Look in your notes or the text.

6. (a) *Answer.* Look in your notes or the text.

   (b) *Answer.* (i) converges to 0, (ii) diverges, and (iii) converges to e.

   (c) *Answer.* Look in your notes or the text.

7. (a) *Answer.* Look in your notes or the text.

   (b) *Answer.* Look in your notes or the text.

   (c) *Answer.* Yeap.