Elementary Functions and Calculus II
Math 132 (Sec 50), Winter 2005
Revision Sheet for the Final Examination

The final will be based on the material we have covered in lectures this quarter. Also remember that material from Math 131 will be assumed, so you should remain familiar with that too. Do not forget that this quarter the second mid-term was not at the very end of the quarter, so there will be topics covered in the final that have not been covered in either of the mid-terms. Also remember I make mistakes too, so if you see something which doesn't look write or I use what you think is new notation, chances are I have made a typo. I will *not* be forgiving if you reproduce mistakes I have made - mathematics should make sense, not least to you. Ask me before the exam if you are unsure. Last time several people copied my mistake in some sigma notation.

You should be familiar with the following concepts and be able to recall definitions and major theorems associated to them:

- **Trigonometry.** You should not the definitions of the trigonometric functions and the basic identities (those which I asked to memorise for Mid-term 1). Be familiar with the others and how they are derived from the basic ones, but if you are required to prove or use them, you will be guided through or told what they are. Recall how to differentiate the trigonometric functions and functions involving trigonometric functions. Also remember the limits $\lim_{x\to 0}\frac{\sin x}{x}$ and $\lim_{x\to 0}\frac{\cos x-1}{x}$ and there uses.

- **Applications of the derivative.** Recall the definitions of maxima and minima and those related to these ideas. Recall the theorem which gives us the existence of maxima and minima. Recall the mean value theorem (for derivatives) and its applications.

- **Sigma notation and sums.** Know the notation $\sum_{k=1}^{n} a_k$ for numbers $a_1, a_2, \ldots, a_n$ and understand what is going on when we express a_k by an expression involving k, e.g. the Riemann sum. Recall the same formulae as in Mid-term 2 and the ideas involved in deriving certain types of formulae (e.g. $\sum_{k=1}^{n} k$).

- **The definite (Riemann) integral and anti-derivatives.** Recall what an anti-derivative is and how two anti-derivatives of a function are related. Know the notation $A_x(f(x))$ where f is a function. Know what a Riemann sum is and the definition of the (Riemann) integral. Recall the first and second fundamental theorems of Calculus. Recall the mean value theorem for integrals. Know how to apply these theorems to problems done in class and in the homework. Know how to calculate volumes of solids of revolution and their (slightly hand-wavy) justification.

- **Transcendental functions** Recall the definition of the natural logarithm. Understand what is an inverse f^{-1} to a function f, and recall the theorem that gives us conditions for the existence of an inverse. Recall the definition of the natural exponential function. Recall the basic rules we proved for manipulating both the natural logarithm and exponential function. Recall the formula we derived for the function exp as a power of the number e, and the definition this lead us to make. Recall the definition of the number e.

Please remember to read carefully what a question is asking of you. You should always try to justify you steps when they are not immediately obvious, and doing so will convince me you understand the material well, which will make my happy, and so increase your mark.

The following questions are in the style you can expect in the test. It would be good preparation for the exam to try these questions. It is also valuable to check you have answered them correctly.

1. (a) Draw a unit circle centred at the origin. On it label the distances t, x and y which are used to define the expressions $x = \cos t$ and $y = \sin t$.

   (b) Using the identity $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$ to prove a formula for $\cos(2t)$ in terms of $\cos t$ and $\sin t$ only.

   (c) Write down (do not prove) what the derivative of the function $\sin : \mathbf{R} \to \mathbf{R}$ is.

   (d) Use part (c) to calculate the derivative of the function $x \mapsto \sin(x^2 + 3x)$.

2. (a) Define what it means to say a function $f : \mathbf{R} \to \mathbf{R}$ has a (global) maximum value M. Define what is means for a function $f : \mathbf{R} \to \mathbf{R}$ to be (strictly) increasing.

   (b) State a theorem which gives us the existence of such a maximum value. State carefully the conditions you require.

   (c) Stating clearly any facts you have learnt in class which you require, find the maximum value of the function $f : \mathbf{R} \to \mathbf{R}$ defined by the formula $f(x) = 2x - x^2$. Stating clearly any more facts you need show that it is indeed a maximum value.

3. (a) State the mean value theorem (for derivatives).

   (b) State the mean value theorem for integrals.

4. (a) Define the function $\ln : (0, \infty) \to \mathbf{R}$.

   (b) State the first fundamental theorem of Calculus.

   (c) Use the first fundamental theorem of Calculus to prove $(\ln)'(x) = \frac{1}{x}$ for all $x \in (0, \infty)$.

   (d) State (do not prove) for formula for expressing $\ln(ab)$ in terms of $\ln a$ and $\ln b$.

Remember, the best preparation the night before the exam is to sleep well. I know this might be hard to do, but a well rested mind thinks more clearly. Good luck.

Solutions to the practice problems

1. (a) Draw a unit circle centred at the origin. On it label the distances t, x and y which are used to define the expressions $x = \cos t$ and $y = \sin t$.

   Answer. Oh.

   (b) Using the identity $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$ to prove a formula for $\cos(2t)$ in terms of $\cos t$ and $\sin t$ only.

   Answer. We use the identity given with $t = \theta = \phi$, then

$$\cos(2t) = \cos(t + t) = \cos t \cos t - \sin t \sin t = \cos^2 t - \sin^2 t$$

   (c) Write down (do not prove) what the derivative of the function $\sin : \mathbf{R} \to \mathbf{R}$ is.

   Answer. It's the function cos.

   (d) Use part (c) to calculate the derivative of the function $x \mapsto \sin(x^2 + 3x)$.

   Answer. Using the chain rule we have

$$\frac{d}{dx}\sin(x^2 + 3x) = (2x + 3)\cos(x^2 + 3x).$$

2. (a) Define what it means to say a function $f : \mathbf{R} \to \mathbf{R}$ has a (global) maximum value M. Define what is means for a function $f : \mathbf{R} \to \mathbf{R}$ to be (strictly) increasing.

   Answer. We say the f as above has a (global) maximum M if $f(x) \leq f(x_0)$ for some fixed x_0 and all $x \in \mathbf{R}$. $f(x_0) = M$. We say f is (strictly) increasing if for $x < y$, we have $f(x) < f(y)$.

   (b) State a theorem which gives us the existence of such a maximum value. State carefully the conditions you require.

   Answer. Let f be a continuous function defined on a closed interval I (i.e. $f : I \to \mathbf{R}$). Then f has a maximum value.

   (c) Stating clearly any facts you have learnt in class which you require, find the maximum value of the function $f : \mathbf{R} \to \mathbf{R}$ defined by the formula $f(x) = 2x - x^2$. Stating clearly any more facts you need show that it is indeed a maximum value.

   Answer. Maximum values can only take place at critical points. Since there are no end points and the function is differentiable, if it exists it is attained where $f'(x) = 0$. We have $f'(x) = 2 - 2x$. This is zero only for $x = 1$. So we can only have a maximum at $x = 1$, then $M = f(1) = 1$.

We have $f'(x) < 0$ for $x > 1$ so it is decreasing to the right of $x = 1$ and $f'(x) > 0$ for $x < 1$, so it is increasing to the left of $x = 1$. Thus $f(1)$ must be a (global) maximum.

3. (a) State the mean value theorem (for derivatives).

   Answer. See notes or text book.

   (b) State the mean value theorem for integrals.

   Answer. See notes or text book.

4. (a) Define the function $\ln : (0, \infty) \to \mathbf{R}$.

   Answer. The function $\ln : (0, \infty) \to \mathbf{R}$ is defined by

$$\ln x = \int_1^x \frac{1}{t}\, dt$$

   for all $x \in (0, \infty)$.

   (b) State the first fundamental theorem of Calculus.

   Answer. See notes or text book.

   (c) Use the first fundamental theorem of Calculus to prove $(\ln)'(x) = \frac{1}{x}$ for all $x \in (0, \infty)$.

   Answer. We have

$$\frac{d}{dx} \int_1^x \frac{1}{t}\, dt = \frac{1}{x}$$

   for all $x \in (0, \infty)$.

   (d) State (do not prove) for formula for expressing $\ln(ab)$ in terms of $\ln a$ and $\ln b$.

   Answer.

$$\ln(ab) = \ln a + \ln b$$