Name: ______________________________

Elementary Functions and Calculus II
Math 132 (Sec 22), Winter 2004
Final Examination
17th March 2004

Instructions: The total time allowed for this examination is 2 hours. The use of notes, textbooks or calculators is prohibited. Write you answer to each question in the space provided below it. Should you require more space write on the reverse of the paper, labeling your answers clearly. Write your name at the top of this sheet. The maximum number of points available for each question or part question is shown in parentheses next to the question. There are 7 questions. You should attempt as many of the questions as you can. Partial credit may be given for incomplete answers.

(7 pts) 1. (a) For a function $f : [a, b] \to \mathbf{R}$, a partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$ and a set of sample points $S = \{\overline{x}_1, \ldots, \overline{x}_n\}$ subordinate to P, define the Riemann sum R_P of f with respect to P and S.

(7 pts) (b) Calculate the Riemann sum R_P where $f : [0, 1] \to \mathbf{R}$ is given by $f(x) = x$, $P = \{0, \frac{1}{3}, 1\}$ and $S = \{\frac{1}{6}, \frac{2}{3}\}$.

(6 pts) (c) For $a < b$ and $f : [a, b] \to \mathbf{R}$, define

$$\int_a^b f(x)\, dx.$$

(5 pts) 2. (a) Please indicate which of the following identities are correct by writing true or false next to the expression.

- $\sin^2 x - \cos^2 \equiv 1$
- $\cos(x+y) \equiv \cos x \cos y - \sin x \sin y$
- $\cos(x+y) \equiv \cos x \cos y + \sin x \sin y$

(5 pts) (b) Recall the values of the following limits.

$$\lim_{x\to 0} \frac{\sin x}{x} =$$
$$\lim_{x\to 0} \frac{1-\cos x}{x} =$$

(4 pts) (c) Why is does it <u>not</u> follow from the main limit theorem that the limits in (b) exist?

(6 pts) (d) Use part (b) to calculate

$$\lim_{x\to 0} \frac{\tan x - \sin x}{x}$$

(Hint: $\tan x - \sin x = \tan x(1-\cos x)$)

(6 pts) 3. (a) Define what it means for $f : I \to \mathbf{R}$, where I is an interval, to have a maximum value M.

(7 pts) (b) State the theorem given in class which tells us when we know f has a maximum value. Remember to state the conditions on f.

(7 pts) (c) Give an example of a continuous function which does <u>not</u> have a maximum value. State clearly the domain of the function.

(10 pts) 4. (a) State the mean value theorem for a function $f : [a, b] \to \mathbf{R}$. State clearly the conditions on f.

(10 pts) (b) Can we apply the mean value theorem to $f : [-1, 1] \to \mathbf{R}$ defined below?

$$f(x) = \begin{cases} \frac{(x+1)(x-1)}{x}, & \text{if} \quad x \neq 0; \\ -1, & \text{if} \quad x = 0. \end{cases}$$

Justify you answer.

5. For numbers a and r, let

$$S_n = \sum_{k=0}^{n} ar^k = a + ar + ar^2 + \cdots + ar^n.$$

(5 pts) (a) Show that $S_n - rS_n = a - ar^n$

(5 pts) (b) Use part (a) to derive a formula for $\sum_{k=0}^{n} ar^k$.

(5 pts) (c) Use your formula from part (b) to calculate $\sum_{k=1}^{n} (\frac{1}{2})^k$. Beware: In this sum k runs from 1 to n not from 0 to n.

(5 pts) (d) Calculate $\sum_{k=1}^{n} 1$.

(6 pts) 6. (a) State the definition of the natural logarithm function $\ln : (0, \infty) \to \mathbf{R}$.

(7 pts) (b) Which theorem can we apply to calculate the derivative of ln (just give the name, not the statement). Write down the derivative of ln at x, and for which x your answer is valid.

(7 pts) (c) Use the rules you know for ln to simplify $\ln(\frac{a}{b}) + \ln(\frac{b}{a})$ for $a, b \in (0, \infty)$.

(4 pts) 7. (a) Give a condition which tells us that a function $f : D \to R$, where $R = \{f(x) | x \in D\}$, will have an inverse. Show that the function ln satisfies this condition.

(4 pts) (b) Define the natural exponential function $\exp : \mathbf{R} \to (0, \infty)$.

(4 pts) (c) Define the number e.

(4 pts) (d) Use the rules you know for the natural logarithm to prove that for any rational number r, we have $\exp(r) = e^r$. (Hint: $e^r = \exp(\ln e^r)$.)

(4 pts) (e) Define e^a for an irrational number a.